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Lightlike Hypersurfaces of Semi-Euclidean Spaces Satisfying Curvature Conditions of Semisymmetry Type

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Abstract

In this paper, we investigate lightlike hypersurfaces which are semi-symmetric, Ricci semi-symmetric, parallel or semi-parallel in a semi-Euclidean space. We obtain that every screen conformal lightlike hypersurface of the Minkowski spacetime is semi-symmetric. For higher dimensions, we show that the semi-symmetry condition of a screen conformal lightlike hypersurface reduces to the semi-symmetry condition of a leaf of its screen distribution. We also obtain that semi-symmetric and Ricci semi-symmetric lightlike hypersurfaces are totally geodesic under certain conditions. Moreover, we show that there exist no non-totally geodesic parallel hypersurfaces in a Lorentzian space.

Key Words: Degenerate metric, Screen conformal lightlike hypersurface, Parallel lightlike hypersurface, Semi-symmetric lightlike hypersurface.

1. Introduction

The class of semi-Riemannian manifolds, satisfying the condition

$$\nabla R = 0, \tag{1.1}$$

is a natural generalization of the class of manifolds of constant curvature, where ∇ is the Levi-Civita connection on semi-Riemannian manifold and R is the corresponding

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curvature tensor. For precise definitions of the symbols used, we refer to Section 2.1.

A semi-Riemannian manifold is called semi-symmetric if

$$\mathbf{R} \cdot R = 0, \tag{1.2}$$

where $\mathbf{R}$ is the curvature operator corresponding to R and the $\cdot$ operation is defined in Section 2.1. Semi-symmetric hypersurfaces of Euclidean spaces were classified by Nomizu [15] and a general study of semi-symmetric Riemannian manifolds was made by Szabo [17].

A semi-Riemannian manifold is said to be Ricci semi-symmetric [7], if the following condition is satisfied:

$$\mathbf{R} \cdot Ric = 0. \tag{1.3}$$

It is clear that every semi-symmetric manifold is Ricci semi-symmetric; the converse is not true in general and a brief discussion of this issue is given in Section 2.1.

If a manifold M is immersed into a manifold $\bar{M}$, the immersion is said to be parallel if the second fundamental form is covariantly constant, i.e., $\nabla h = 0$, where ∇ is an affine connection $\bar{M}$ and h is the second fundamental form of the immersion. The general classification of parallel submanifolds of Euclidean space was obtained in [13] by D. Ferus. He showed that such an immersion is an isometric immersion into an n-dimensional symmetric R-space imbedded in R^{n+p} in the standard way. The general theory of lightlike submanifolds was introduced and presented in a book by Duggal-Bejancu [10]. The theory of lightlike submanifolds is a new area of differential geometry and it is very different from Riemannian geometry as well as semi-Riemannian geometry.

In third section of this paper, we consider a lightlike hypersurface of the semi-Euclidean space and study semi-symmetry conditions on this hypersurface. Our main result, in this section, states that every screen conformal lightlike hypersurface (Definition 3) of the Minkowski spacetime R_1^4 is semi-symmetric. For R_q^{n+2},$n \geq 3$ we show that semi-symmetry of a lightlike hypersurface depends on the geometry of a leaf of screen distribution.

In section four, we study Ricci semi-symmetric lightlike hypersurfaces and obtain that Ricci semi-symmetric lightlike hypersurfaces are totally geodesic under a certain condition. In this section, we also obtain that semi-symmetric lightlike hypersurfaces are totally geodesic under a condition in terms of the Ricci tensor.

In section five, we investigate parallel hypersurface of a Lorentzian manifold. In fact, we show that every parallel lightlike hypersurface must be totally geodesic. Then we study semi-parallel lightlike hypersurfaces in a semi-Euclidean space. We note that the semi-parallel hypersurfaces were defined in [8] as a generalization of parallel hypersurfaces for Riemannian case.

2. Preliminaries

In this section, we will give a brief review of curvature conditions of semi-symmetry type and lightlike submanifolds of semi-Riemannian manifolds. A full discussion of the contents of this section can be found in [7] and [10], respectively. In this paper, we will assume that every object in hand is smooth.

2.1. Curvature Conditions of Symmetry Type

Let (M, g) be a semi-Riemannian manifold. We denote its curvature operator by $R(X, Y)$

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for $X, Y \in \Gamma(TM)$, where ∇ denotes the Levi-Civita connection on M. Then the Riemannian Christoffel curvature tensor R and the Ricci tensor Ric are defined by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W), \tag{2.4}$$

$$Ric(X,Y) = trace\{Z \to R(X,Y)Z\}, \tag{2.5}$$

respectively.

For a $(0,k)$-tensor field T on M, $k \geq 1$, the $(0,k+2)$ tensor field $R \cdot T$ is defined by

$$\begin{aligned}(R \cdot T)(X_1, ..., X_k, X, Y) &= -T(R(X,Y)X_1, X_2, ..., X_k) \\ &\quad -... - T(X_1, ..., X_{k-1}, R(X,Y)X_k) \end{aligned}\tag{2.6}$$

for $X, Y, X_1, ..., X_k \in \Gamma(TM)$. Curvature conditions, involving the form $R \cdot T = 0$, are called curvature conditions of semi-symmetric type [7].

A semi-Riemannian manifold M is said to be semi-symmetric if it satisfies the condition $R \cdot R = 0$. Thus, from (2.6) and properties of curvature tensor, we have

$$\begin{aligned}(R(X,Y) \cdot R)(U,V)W &= R(X,Y)R(U,V)W - R(U,V)R(X,Y)W \\ &- R(R(X,Y)U,V)W - R(U,R(X,Y)V)W = 0 \end{aligned}\tag{2.7}$$

for any $X, Y, U, V, W \in \Gamma(TM)$.

A semi-Riemannian manifold M is said to be Ricci semi-symmetric if it satisfies the condition $R \cdot Ric = 0$, i.e.,

$$\begin{aligned}(R(X,Y) \cdot Ric)(X_1, X_2) &= -Ric(R(X,Y)X_1, X_2) \\ &- Ric(X_1, R(X,Y)X_2) = 0, \end{aligned} \tag{2.8}$$

for $X, Y, X_1, X_2 \in \Gamma(TM)$.

In [8], Deprez defined and studied semi-paralel hypersurfaces in Euclidean n space. We recall that a hypersurface M of a semi-Riemannian manifold $\bar{M}$ is said to be semi-parallel if the following condition is satisfied for every point $p \in M$ and every vector fields $X, Y, Z, W \in \Gamma(TM)$:

$$(R(X,Y)h)(Z,W) = -h(R(X,Y)Z,W) - h(Z, R(X,Y)W) = 0, \tag{2.9}$$

where h is the second fundamental form and R is the curvature tensor field of M.

Although conditions (1.2) and (1.3) are not equivalent for manifolds in general, P.J. Ryan [16] raised the following question for hypersurfaces of Euclidean spaces in 1972: "Are the conditions $\mathbf{R} \cdot R = 0$ and $\mathbf{R} \cdot Ric = 0$ equivalent for hypersurfaces of Euclidean spaces?" Although there are many results which contributed to the solution of the above question in the affirmative under some conditions (see [5], [6], [14], [19]), Abdalla and Dillen [1] gave an explicit example of a hypersurface in Euclidean space E^{n+1} $(n \geq 4)$ that is Ricci semi-symmetric but not semi-symmetric (See also [7] for another example.). This result shows that the conditions $\mathbf{R} \cdot R = 0$ and $\mathbf{R} \cdot Ric = 0$ are not equivalent for hypersurfaces of Euclidean space in general. A recent survey on Ricci semi-symmetric spaces and contributions to the solution of above problem can be found in [7]. We note that, in [20], I. Van de Woestijne and L. Verstraelen used the standard forms of a symmetric operator in a Lorentzian vector space to give an algebraic proof that the shape operator of a semisymmetric hypersurface at a point with type number greater than 2 is diagonalizable with exactly two eigenvalues, one of which is zero.

2.2. Lightlike Hypersurfaces

Let $(\bar{M}, \bar{g})$ be an $(m+2)$-dimensional semi-Riemannian manifold with the indefinite metric $\bar{g}$ of index $q \in \{1, ..., m+1\}$ and M be a hypersurface of $\bar{M}$. We denote the tangent space at $x \in M$ by T_xM. Then

$$T_xM^{\perp} = \{V_x \in T_x\bar{M} | \bar{g}_x(V_x, W_x) = 0, \forall W_x \in T_xM\}$$

and

$$RadT_xM = T_xM \cap T_xM^{\perp}.$$

Then, M is called a lightlike hypersurface of $\bar{M}$ if $RadT_xM \neq \{0\}$ for any $x \in M$. Thus $TM^{\perp} = \bigcap_{x\in M} T_xM^{\perp}$ becomes a one- dimensional distribution on M. We denote $F(M)$ the algebra of differential functions on M and by $\Gamma(E)$ the $F(M)$- module of differentiable sections of a vector bundle E over M.

Definition 1. ([10], p:78): *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. A complementary vector subbundle $S(TM)$ to $TM^{\perp}$ in TM is called a screen distribution of M.*

It is known from ([10], Proposition 2.1, p:5) that $S(TM)$ is non-degenerate. Thus, we have the orthogonal direct sum

$$TM = TM^{\perp} \oplus_{\perp} S(TM), \tag{2.10}$$

where $\oplus_{\perp}$ denotes the orthogonal direct sum. From (2.10), we observe that $TM^{\perp}$ lies in the tangent bundle of the lightlike hypersurface M. Thus a vital problem of this theory is to replace the intersecting part by a vector bundle of $T\bar{M}\mid_M$ whose sections are nowhere tangent to M. Next theorem shows that there exists a such complementary (non-orthogonal) vector bundle to M in $T\bar{M}$.

Theorem 2.1. ([10], p: 79): *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $U \subset M$, there exists a unique section N of $tr(TM)$ on U such that*

$$\bar{g}(\xi, N) = 1, \bar{g}(N, N) = \bar{g}(N, X) = 0 \quad \forall X \in \Gamma(S(TM|_U)). \tag{2.11}$$

It follows from (2.11) that $tr(TM)$ is a lightlike vector bundle such that $tr(TM)_x \cap T_xM = \{0\}$ for any $x \in M$. Thus from (2.10) and (2.11) we have

$$\begin{aligned} T\bar{M}|_M &= S(TM) \oplus_{\perp} (TM^{\perp} \oplus tr(TM)) \\ &= TM \oplus tr(TM). \end{aligned} \tag{2.12}$$

Definition 2. ([10], p:79): *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then the complementary (non-orthogonal) vector bundle $tr(TM)$ to the tangent bundle TM in $T\bar{M}\mid_M$ is called the lightlike transversal bundle of M with respect to* *screen distribution* $S(TM)$.

Suppose M is a lightlike hypersurface of $\bar{M}$ and $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (2.12) we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.13}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$, $h(X,Y)$ and $\nabla^t_X V$ belong to $\Gamma(tr(TM))$. We note that it is easy to see that ∇ is a torsion free connection, h is a $tr(TM)$ valued, symmetric $F(M)-$ bilinear form on TM, A_V is a $F(M)-$ linear operator on $\Gamma(TM)$ and ∇^t is a linear connection on $tr(TM)$. h and A_V are called the second fundamental form and shape operator of the lightlike hypersurface M, respectively.

Locally suppose $\{\xi, N\}$ is a pair of vector fields on U in Theorem 2.1. Then we define a symmetric bilinear form B and $1-$ form τ on U by

$$B(X,Y) = \bar{g}(h(X,Y),\xi) \quad \text{and} \quad \tau(X) = \bar{g}(\nabla^t_X N, \xi)$$

for $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^{\perp})$ and $N \in \Gamma(tr(TM))$. Thus (2.13) and (2.14) become

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N \tag{2.15}$$

and

$$\bar{\nabla}_X N = -A_N X + \tau(X)N \tag{2.16}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$.

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.10). We obtain

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi \tag{2.17}$$

and

$$\nabla_X \xi = -A^*_\xi X + \upsilon(X)\xi \tag{2.18}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla^*_X PY, A^*_\xi X \in \Gamma(S(TM))$ and C is a $1-$ form on U defined by

$$C(X, PY) = \bar{g}(\nabla_X PY, N) \tag{2.19}$$

for $X, Y \in \Gamma(TM)$. C and A^* are called the second fundamental form and shape operator of the screen distribution $S(TM)$, respectively. From (2.11), (2.15),(2.16) and (2.18) we obtain $\upsilon(X) = -\tau(X)$, thus (2.18) becomes

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi. \tag{2.20}$$

By direct calculations, using (2.15),(2.16), (2.17) and (2.20) we obtain the following lemma.

Lemma 2.1. ([10], p:85) *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then we have*

$$g(A_N Y, PW) = C(Y, PW), \quad g(A_N Y, N) = 0 \tag{2.21}$$

$$g(A^*_\xi X, PY) = B(X, PY) \tag{2.22}$$

for $X, Y, W \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM)$.

We note that the second equation of (2.21) implies that $A_N X \in \Gamma(S(TM))$ for $X \in \Gamma(TM)$, i.e., A_N is $\Gamma(S(TM))-$ valued. On the other hand, from $\bar{g}(\bar{\nabla}_X \xi, \xi) = 0$ we have

$$B(X, \xi) = 0. \tag{2.23}$$

We now recall the definition of screen conformal lightlike hypersurfaces of a semi-Riemannian manifold $\bar{M}$.

Definition 3. [2]. *A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold is screen conformal if the shape operators A_N and A^*_ξ of M and its screen distribution $S(TM)$ are related by*

$$A_N = \varphi A^*_\xi, \tag{2.24}$$

where φ is a non-vanishing smooth function on a neighborhood U in M. In case $U = M$ the screen conformality is said to be global.

We note that there are many examples of screen conformal lightlike hypersurfaces of semi-Riemannian manifolds. Next, we give two examples of screen conformal lightlike hypersurfaces of semi-Euclidean spaces; for more examples, see [2].

Examples.(1) The Lightlike Cone $\bigwedge_0^3$ of R_1^4: Let R_1^4 be the space R^4 endowed with the semi-Euclidean metric

$$\bar{g}(x,y) = -x^1y^1 + x^2\,y^2 + x^3y^3 + x^4\,y^4,\ x = \sum_{i=1}^{4} x^i\,\frac{\partial}{\partial\,x^i}.$$

The lightlike cone is given by the equation $-(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 0, x \neq 0$. It is known that the lightlike cone is a screen conformal lightlike hypersurface [2].

(2) Lightlike Monge Hypersurfaces of R_1^4: Let D be an open set of R_1^4 and $F : D \to R$ be a smooth function on D. Then the set

$$M = \{(x^1, x^2, x^3, x^4) \in R^4 :\, x^1 = F(x^2, x^3, x^4)\}$$

is called a Monge hypersurface. A Monge hypersurface of R_1^4 is lightlike if and only if F is a solution of the partial differential equation

$$1 + (\frac{\partial\,F}{\partial x_1})^2 = (\frac{\partial\,F}{\partial x_2})^2 + (\frac{\partial\,F}{\partial x_3})^2 + (\frac{\partial\,F}{\partial x_4})^2.$$

It is known that a lightlike Monge hypersurface is screen conformal [2].

3. Semi-symmetric Lightlike Hypersurfaces in Semi-Euclidean Spaces

In this section, we consider semi-symmetric lightlike hypersurfaces in a semi-Euclidean space. First, we give the Gauss equation for a lightlike hypersurface of a semi-Euclidean space $R_q^{(n+2)}$. Then we show that every screen conformal lightlike hypersurface of the Minkowski spacetime is semi-symmetric. For higher dimensions, we show that the semi-symmetry condition of a screen conformal lightlike hypersurface M has close relation with the semi-symmetry condition of a leaf of its screen distribution. From now on, we denote a lightlike hypersurface by M and use A for A_N.

Proposition 3.1. *Let M be a lightlike hypersurface of a semi-Euclidean space $R_q^{(n+2)}$. Then the Gauss equation of M is given by*

$$R(X,Y)Z = B(Y,Z)AX - B(X,Z)AY \tag{3.1}$$

for any $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$.

Proof. For a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$, from ([10], p:93) we have

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ &+ (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \end{aligned} \tag{3.2}$$

where $\bar{R}$ and R are curvature tensor fields of $\bar{M}$ and M, respectively. We note that $(\nabla_X h)(Y,Z)$ is defined by

$$(\nabla_X h)(Y,Z) = \nabla_X^t h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{3.3}$$

By assumption, $\bar{M} = R_q^{(n+2)}$ is a semi-Euclidean space, hence $\bar{R} = 0$. Then (3.2) becomes

$$R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0.$$

On the other hand, (2.13) and (2.15) imply that $h(X,Y) = B(X,Y)N$ for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$. Thus, we get

$$R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0.$$

Then comparing the tangential and transversal parts of the above equation, we obtain (3.1).

We note that $g(R(X,Y)Z,W) \neq -g(R(X,Y)W,Z)$, $\forall X,Y,Z,W \in \Gamma(TM)$, for a lightlike hypersurface in general.

Definition 4. *Let M be a lightlike hypersurface of a semi-Euclidean space. We say that M is a semi-symmetric if the following condition is satisfied*

$$(R(X,Y) \cdot R)(X_1, X_2, X_3, X_4) = 0 \tag{3.4}$$

for $X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)$.

Notice that it is easy to see that

$$(R(X,Y)\cdot R)(X_1,X_2,X_3,\xi)=0$$

for $\xi\in\Gamma(TM^{\perp})$. Thus the condition (3.4) is equivalent to the following condition

$$(R(X,Y)\cdot R)(X_1,X_2,X_3,PX_4)=0 \tag{3.5}$$

for $X,Y,X_1,X_2,X_3,X_4\in\Gamma(TM)$. We also note that (3.4) and (3.5) do not imply the equation (2.7) due to $g(R(X,Y)Z,W)\neq -g(R(X,Y)W,Z)$ in general, for $X,Y,Z,W\in\Gamma(TM)$.

Now, from (3.5) and (3.1), we obtain

$$\begin{aligned}(R(X,Y)\cdot R)(X_1,X_2,X_3,PX_4) &= B(Y,X_1)[B(AX,X_3)g(AX_2,PX_4)\\ &- B(X_2,X_3)g(A^2X,PX_4)]+B(X,X_1)[B(X_2,X_3)g(A^2Y,PX_4)\\ &- B(AY,X_3)g(AX_2,PX_4)]+g(AX_1,PX_4)[-B(Y,X_2)B(AX,X_3)\\ &+ B(X,X_2)B(AY,X_3)]+B(X_1,X_3)[B(Y,X_2)g(A^2X,PX_4)\\ &- B(X,X_2)g(A^2Y,PX_4)]+g(AX_1,PX_4)[-B(X_3,Y)B(X_2,AX)\\ &+ B(X,X_3)B(X_2,AY)]+g(AX_2,PX_4)[B(X_3,Y)B(X_1,AX)\\ &- B(X,X_3)B(X_1,AY)]+B(X_2,X_3)[-B(Y,X_4)g(AX_1,AX)\\ &+ B(X,PX_4)g(AX_1,AY)]+B(X_1,X_3)[B(Y,PX_4)g(AX_2,AX)\\ &- B(X,PX_4)g(AX_2,AY)]\end{aligned} \tag{3.6}$$

for any $X,Y,X_1,X_2,X_3,X_4\in\Gamma(TM)$.

Proposition 3.2. *Every screen conformal lightlike hypersurface of the Minkowski spacetime is a semi-symmetric lightlike hypersurface.*

Proof. First, from (3.6), we have

$$
\begin{aligned}
(R(X,Y)\cdot R)\,(\xi,X_2,X_3,PX_4) = {} & B(Y,\xi)[B(AX,X_3)g(AX_2,PX_4) \\
& -\ B(X_2,X_3)g(A^2X,PX_4)] \\
& +\ B(X,\xi)[B(X_2,X_3)g(A^2Y,PX_4) - B(AY,X_3)g(AX_2,PX_4)] \\
& +\ g(A\xi,PX_4)[-B(Y,X_2)B(AX,X_3) + B(X,X_2)B(AY,X_3)] \\
& +\ B(\xi,X_3)[B(Y,X_2)g(A^2X,PX_4) - B(X,X_2)g(A^2Y,PX_4)] \\
& +\ g(A\xi,PX_4)[-B(X_3,Y)B(X_2,AX) + B(X,X_3)B(X_2,AY)] \\
& +\ g(AX_2,PX_4)[B(X_3,Y)B(\xi,AX) - B(X,X_3)B(\xi,AY)] \\
& +\ B(X_2,X_3)[-B(Y,PX_4)B(A\xi,AX + B(X,PX_4)g(A\xi,AY)] \\
& +\ B(\xi,X_3)[B(Y,PX_4)g(AX_2,AX) - B(X,PX_4)g(AX_2,AY)]
\end{aligned}
$$

for any $X, Y, X_2, X_3, X_4 \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. Then, from (2.23), we get

$$
\begin{aligned}
(R(X,Y)\cdot R)\,(\xi,X_2,X_3,PX_4) = {} & g(A\xi,PX_4)[-B(Y,X_2)B(AX,X_3) \\
& +\ B(X,X_2)B(AY,X_3)] \\
& +\ g(A\xi,PX_4)[-B(X_3,Y)B(X_2,AX) + B(X,X_3)B(X_2,AY)] \\
& +\ B(X_2,X_3)[-B(Y,PX_4)B(A\xi,AX + B(X,PX_4)g(A\xi,AY)].
\end{aligned}
$$

Then, (2.24) implies that

$$
\begin{aligned}
(R(X,Y)\cdot R)\,(\xi,X_2,X_3,PX_4) = {} & \varphi g(A^*_\xi\xi,PX_4)[-B(Y,X_2)B(AX,X_3) \\
& +\ B(X,X_2)B(AY,X_3)] \\
& +\ \varphi g(A^*_\xi\xi,PX_4)[-B(X_3,Y)B(X_2,AX) + B(X,X_3)B(X_2,AY)] \\
& +\ \varphi B(X_2,X_3)[-B(Y,PX_4)B(A^*_\xi\xi,AX + B(X,PX_4)g(A^*_\xi\xi,AY)].
\end{aligned}
$$

From (2.22) and (2.23), we have $A^*_\xi\xi = 0$. Thus, we derive

$$(R(X,Y)\cdot R)(\xi,X_2,X_3,PX_4) = 0.$$

In a similar way, we obtain

$$(R(X,Y)\cdot R)(X_1,X_2,\xi,PX_4) = 0, (R(\xi,Y)\cdot R)(X_1,X_2,X_3,PX_4) = 0$$

and

$$(R(X,Y)\cdot R)(X_1,\xi,X_3,PX_4) = 0, (R(X,\xi)\cdot R)(X_1,X_2,X_3,PX_4) = 0.$$

for $X_1, X_2, X_3, X_4 \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$. Let $\{X_1, X_2, \xi, N\}$ be a quasi-orthonormal basis of R^4_1 such that $S(TM) = span\{X_1, X_2\}$ and $tr(TM) = span\{N\}$. From (3.6), we have

$$\begin{aligned}
(R(X_1, X_2) \cdot R)\,(X_1, X_2, X_1, X_2) = B(X_2, X_1)[&B(AX_1, X_1)g(AX_2, X_2)\\
-\ & B(X_2, X_1)g(A^2X_1, PX_2)]\\
+\ & B(X_1, X_1)[B(X_2, X_1)g(A^2X_2, X_2) - B(AX_2, X_31)g(AX_2, PX_2)]\\
+\ & g(AX_1, X_2)[-B(X_2, X_2)B(AX_1, X_1) + B(X_1, X_2)B(AX_2, X_1)]\\
+\ & B(X_1, X_1)[B(X_2, X_2)g(A^2X_1, X_2) - B(X_1, X_2)g(A^2X_2, X_2)]\\
+\ & g(AX_1, X_2)[-B(X_1, X_2)B(X_2, AX_1) + B(X_1, X_1)B(X_2, AX_2)]\\
+\ & g(AX_2, X_2)[B(X_1, X_2)B(X_1, AX_1) - B(X_1, X_1)B(X_1, AX_2)]\\
+\ & B(X_2, X_1)[-B(X_2, X_2)B(AX_1, AX_1 + B(X_1, X_2)g(AX_1, AX_2)]\\
+\ & B(X_1, X_1)[B(X_2, X_2)g(AX_2, AX_1) - B(X_1, X_2)g(AX_2, AX_2)].
\end{aligned}$$

Since $A_N X \in \Gamma(S(TM))$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$ and $A = A_N$ is self-adjoint on $S(TM)$, we get

$$\begin{aligned}
(R(X_1, X_2) \cdot R)\,(X_1, X_2, X_1, X_2) = B(X_2, X_1)[&B(AX_1, X_1)g(AX_2, X_2)\\
-\ & B(X_2, X_1)g(AX_1, AX_2)]\\
+\ & B(X_1, X_1)[B(X_2, X_1)g(AX_2, AX_2) - B(AX_2, X_1)g(AX_2, X_2)]\\
+\ & g(AX_1, X_2)[-B(X_2, X_2)B(AX_1, X_1) + B(X_1, X_2)B(AX_2, X_1)]\\
+\ & B(X_1, X_1)[B(X_2, X_2)g(AX_1, AX_2) - B(X_1, X_2)g(AX_2, AX_2)]\\
+\ & g(AX_1, X_2)[-B(X_1, X_2)B(X_2, AX_1) + B(X_1, X_1)B(X_2, AX_2)]\\
+\ & g(AX_2, X_2)[B(X_1, X_2)B(X_1, AX_1) - B(X_1, X_1)B(X_1, AX_2)]\\
+\ & B(X_2, X_1)[-B(X_2, X_2)g(AX_1, AX_1 + B(X_1, X_2)g(AX_1, AX_2)]\\
+\ & B(X_1, X_1)[B(X_2, X_2)g(AX_2, AX_1) - B(X_1, X_2)g(AX_2, AX_2)].
\end{aligned}$$

Then, using (2.24), we arrive at

$$\begin{aligned}(R(X_1,X_2)\cdot R)\,(X_1,X_2,X_1,X_2) &= \varphi B(X_2,X_1)[B(AX_1,X_1)g(A^*_\xi X_2,X_2)\\ &- B(X_2,X_1)g(A^*_\xi X_1,AX_2)]\\ &+ \varphi B(X_1,X_1)[B(X_2,X_1)g(A^*_\xi X_2,AX_2)-B(AX_2,X_1)g(A^*_\xi X_2,X_2)]\\ &+ \varphi g(A^*_\xi X_1,X_2)[-B(X_2,X_2)B(AX_1,X_1)+B(X_1,X_2)B(AX_2,X_1)]\\ &+ \varphi B(X_1,X_1)[B(X_2,X_2)g(A^*\xi X_1,AX_2)-B(X_1,X_2)g(A^*_\xi X_2,AX_2)]\\ &+ \varphi g(A^*_\xi X_1,X_2)[-B(X_1,X_2)B(X_2,AX_1)+B(X_1,X_1)B(X_2,AX_2)]\\ &+ \varphi g(A^*_\xi X_2,X_2)[B(X_1,X_2)B(X_1,AX_1)-B(X_1,X_1)B(X_1,AX_2)]\\ &+ \varphi B(X_2,X_1)[-B(X_2,X_2)g(A^*_\xi X_1,AX_1+B(X_1,X_2)g(A^*_\xi X_1,AX_2)]\\ &+ \varphi B(X_1,X_1)[B(X_2,X_2)g(A^*_\xi X_2,AX_1)-B(X_1,X_2)g(A^*_\xi X_2,AX_2)].\end{aligned}$$

Thus, using (2.22), we obtain

$$\begin{aligned}(R(X_1,X_2)\cdot R)\,(X_1,X_2,X_1,X_2) &= \varphi B(X_2,X_1)[B(AX_1,X_1)B(X_2,X_2)\\ &- B(X_2,X_1)B(X_1,AX_2)]\\ &+ \varphi B(X_1,X_1)[B(X_2,X_1)B(X_2,AX_2)-B(AX_2,X_1)B(X_2,X_2)]\\ &+ \varphi B(X_1,X_2)[-B(X_2,X_2)B(AX_1,X_1)+B(X_1,X_2)B(AX_2,X_1)]\\ &+ \varphi B(X_1,X_1)[B(X_2,X_2)B(X_1,AX_2)-B(X_1,X_2)B(X_2,AX_2)]\\ &+ \varphi B(X_1,X_2)[-B(X_1,X_2)B(X_2,AX_1)+B(X_1,X_1)B(X_2,AX_2)]\\ &+ \varphi B(X_2,X_2)[B(X_1,X_2)B(X_1,AX_1)-B(X_1,X_1)B(X_1,AX_2)]\\ &+ \varphi B(X_2,X_1)[-B(X_2,X_2)B(X_1,AX_1+B(X_1,X_2)B(X_1,AX_2)]\\ &+ \varphi B(X_1,X_1)[B(X_2,X_2)B(X_2,AX_1)-B(X_1,X_2)B(X_2,AX_2)].\end{aligned}$$

Since B is symmetric, by direct computations, we get

$$\begin{aligned}(R(X_1,X_2)\cdot R)\,(X_1,X_2,X_1,X_2) &= \varphi\{(B(X_2,X_1))^2B(X_1,AX_2)\\ &- (B(X_1,X_2))^2B(X_2,AX_1)\\ &- B(X_2,X_2)B(X_1,X_1)B(X_1,AX_2)\\ &+ B(X_1,X_1)B(X_2,X_2)B(X_2,AX_1)\}. \qquad (3.7)\end{aligned}$$

On the other hand, from (2.22) and (2.24), we have

$$B(AX_2,X_1)=g(A^*_\xi X_1,AX_2)=g(\varphi A^*_\xi X_1,A^*_\xi X_2)=g(AX_1,A^*_\xi X_2).$$

Thus, using again (2.22), we get

$$B(AX_2, X_1) = B(X_2, AX_1). \tag{3.8}$$

Then, from (3.7) and (3.8), we obtain

$$(R(X_1, X_2) \cdot R)(X_1, X_2, X_1, X_2) = 0.$$

In a similar way, we have

$$\begin{aligned} (R(X_1, X_2) \cdot R)(X_1, X_1, X_2, X_2) &= (R(X_1, X_2) \cdot R)(X_2, X_1, X_1, X_2) = 0, \\ (R(X_1, X_2) \cdot R)(X_2, X_1, X_2, X_1) &= (R(X_1, X_2) \cdot R)(X_2, X_2, X_1, X_1) = 0. \end{aligned}$$

and

$$(R(X_1, X_2).R)(X_1, X_2, X_2, X_1) = 0.$$

Thus proof is complete. □

Remark 1. From Proposition 3.2, it follows that lightlike cone of R^4_1, lightlike Monge hypersurface of R^4_1 and lightlike surfaces of R^3_1 are examples of semi-symmetric lightlike hypersurfaces. We also note that Proposition 3.1 is valid for a semi-Euclidean space R^4_q, $1 \leq q < 4$.

Let M be a screen conformal lightlike hypersurface of an $(n+2)$ dimensional semi-Euclidean space. Then, it is known that the screen distribution of M is integrable [2]. We denote a leaf of the screen distribution by M'. Then, we have the following theorem.

Theorem 3.1. *Let M be a screen conformal lightlike hypersurface of an $(n+2)$ dimensional semi-Euclidean space,$n \geq 3$. Then M is semi-symmetric if and only if any leaf M' of $S(TM)$ is semi-symmetric in semi-Euclidean space, that is, the curvature tensor of M' satisfies the condition (2.7) in semi-Euclidean space.*

Proof. Using (3.1) and (2.24) we obtain

$$g(R(X, Y)PZ, PW) = \varphi\{B(Y, Z)B(X, PW) - B(X, Z)B(Y, PW)\} \tag{3.9}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Then, by straightforward computations, using (2.17), (2.20),(2.21),(2.23) and (2.24), we get

$$\begin{aligned} g(R(X,Y)PZ, PW) &= g(R^*(X,Y)PZ, PW) - \varphi\{B(Y,PZ)B(X,PW) \\ &+ B(X,PZ)B(Y,PW)\} \end{aligned} \tag{3.10}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Thus, from (3.9) and (3.10), we derive

$$g(R(X,Y)PZ, PW) = \frac{1}{1+\varphi} g(R^*(X,Y)PZ, PW) \tag{3.11}$$

On the other hand, from (2.21) and (3.1), we get

$$g(R(X,Y)Z, N) = 0, \forall X, Y, Z \in \Gamma(TM), N \in \Gamma(tr(TM)). \tag{3.12}$$

Thus, from (3.11) and (3.12), we conclude that

$$R(X,Y)PZ = \frac{1}{1+\varphi} R^*(X,Y)PZ \tag{3.13}$$

Hence, using algebraic properties of the curvature tensor field, we have

$$(R(X,Y)\cdot R)(U,V,W,Z) = \frac{1}{(1+\varphi)^2}(R^*(X,Y)\cdot R^*)(U,V,W,Z) \tag{3.14}$$

for any $X, Y, U, V, W \in \Gamma(S(TM))$. Thus the proof is complete. □

Remark 2. The above theorem shows us that the semi-symmetry of a screen conformal lightlike hypersurface of an $(n+2)$ semi-Euclidean space is related with the semi-symmetry of a leaf M' of its integrable screen distribution. In Lorentzian case, since screen distribution is Riemannian, studying semi-symmetry of a screen conformal lightlike hypersurface is exactly same with a Riemannian manifold. In fact, we can see from proof of Theorem 3.1. the curvature conditions of a screen conformal lightlike hypersurface reduces to the curvature conditions of a leaf of its screen distribution.

4. Ricci Semi-symmetric Lightlike Hypersurfaces in Semi-Euclidean Spaces

In this section, we study Ricci semi-symmetric lightlike hypersurfaces of semi-Euclidean spaces and obtain that Ricci semi-symmetric lightlike hypersurfaces are totally geodesic under a condition. We also give a theorem on semi-symmetric lightlike hypersurfaces of semi-Euclidean spaces in terms of the Ricci tensor. First, we need the expression of the Ricci tensor of a lightlike hypersurface.

Lemma 4.1. *Let M be a lightlike hypersurface of semi-Euclidean $(n+2)$ space. Then the Ricci tensor Ric of M is given by*

$$Ric(X,Y) = -\sum_{i=1}^{n} \epsilon_i \{B(X,Y)C(w_i, wi)\} - g(A_\xi^* Y, AX) \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$, where $\epsilon_i = \pm 1$ and $\{w_i\}_{i=1}^n$ is an orthonormal basis of $S(TM)$

Proof. The Ricci tensor of a lightlike hypersurface is given by

$$Ric(X,Y) = \sum_{i=1}^{n} \epsilon_i g(R(X,w_i)Y, w_i) - \bar{g}(R(X,\xi)Y, N)\}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^{\perp}$ and $N \in \Gamma(tr(TM)$, where $\{w_i\}_{i=1}^n$ is a basis of $S(TM)$. Then, from (2.21) and (3.1), we have

$$Ric(X,Y) = -\sum_{i=1}^{n} \epsilon_i \{B(X,Y)C(w_i, wi) - B(Y, w_i)C(X, w_i).$$

Using (2.21) and (2.22), we get

$$Ric(X,Y) = -\sum_{i=1}^{n} \epsilon_i \{B(X,Y)C(w_i, wi)\} - g(\sum_{i=1}^{n} \epsilon_i g(A_\xi^* Y, w_i)w_i, AX).$$

Hence, we have (4.1). □

Definition 5. *Let M be a lightlike hypersurface of a semi-Euclidean space. Then we say that M is Ricci semi-symmetric if the following condition is satisfied*

$$(R(X,Y) \cdot Ric)(X_1, X_2) = 0 \tag{4.2}$$

for $X, Y, X_1, X_2 \in \Gamma(TM)$.

Next we give a theorem which shows the effect of Ricci semi-symmetric condition on the geometry of lightlike hypersurfaces of a semi-Euclidean space.

Theorem 4.1. *Let M be a Ricci semi-symmetric lightlike hypersurface of an $(n+2)$-dimensional semi-Euclidean space. Then either M is totally geodesic or $Ric(\xi, A\xi) = 0$ for $\xi \in \Gamma(TM^{\perp})$,where Ric is the Ricci tensor of M and A denotes the shape operator defined in (2.16)*

Proof. From (3.1), (2.8) and (4.2), we obtain

$$\begin{aligned}(R(X,Y)\cdot Ric)(X_1,X_2) &= \alpha\{-B(X,X_1)B(AY,X_2)+B(Y,X_1)B(AX,X_2)\\ &- B(X,X_2)B(X_1,AY)+B(Y,X_2)B(X_1,AX)\}\\ &- B(X,X_1)B(X_2,A^2Y)+B(Y,X_1)B(X_2,A^2X)\\ &- B(X,X_2)B(AY,AX_1)+B(Y,X_2)B(AX,AX_1)\end{aligned}$$

for $X, Y, X_1, X_2 \in \Gamma(TM)$, where $\alpha = \sum_{i=1}^{n} \epsilon_i C(w_i, wi)$. Now, suppose that M is Ricci semi-symmetric lightlike hypersurface. Taking $X_1 = \xi$ in the above equation and using (2.23), we obtain

$$-B(X,X_2)B(AY,A\xi)+B(Y,X_2)B(AX,A\xi)=0.$$

Hence for $Y = \xi$ we derive

$$B(X,X_2)B(A\xi,A\xi)=0.$$

So, if $B(X, X_2) = 0$ for any $X, X_2 \in \Gamma(TM)$, then M is totally geodesic. If M is not totally geodesic, it follows that $B(A\xi, A\xi) = 0$, then from (4.1) we obtain $Ric(\xi, A\xi) = 0$. □

Theorem 4.2. *Let M be a lightlike hypersurface of a semi-Euclidean $(n+2)$ space such that $Ric(\xi, X) = 0$, $\forall X \in \Gamma(TM)$, $\xi \in \Gamma(TM^{\perp}$ and $A\xi$ is a non-null vector field. Then M is semi-symmetric if and only if M is totally geodesic, where Ric is the Ricci tensor of M and A is the shape operator of M.*

Proof. Suppose that M is a semi-symmetric lightlike hypersurface of a semi-Euclidean

$(n+2)$ space. Taking $X_1 = \xi$ in (3.6), we obtain

$$\begin{aligned}
&\{-B(Y,X_2)B(AX,X_3) \ + \ B(X,X_2)B(AY,X_3)\}g(A\xi,PX_4)\\
&\{-B(X_3,Y)B(X_2,AX) \ + \ B(X,X_3)B(X_2,AY)\}g(A\xi,PX_4)\\
&\{-B(Y,PX_4)g(A\xi,AX) \ + \ B(X,PX_4)g(A\xi,AY)\}B(X_2,X_3) = 0.
\end{aligned}$$

Then, for $Y = \xi$, we have

$$\begin{aligned}
B(X,X_2)B(A\xi,X_3)g(A\xi,PX_4) \ &+ \ B(X,X_3)B(X_2,A\xi)g(A\xi,PX_4)\\
&+ \ B(X,PX_4)g(A\xi,A\xi)B(X_2,X_3) = 0.
\end{aligned}$$

Thus, by assumption, $R(\xi, X) = 0$, we have $B(X, A\xi) = 0$. Hence, we get

$$B(X,PX_4)g(A\xi,A\xi)B(X_2,X_3) = 0.$$

Since $A\xi$ is a non-null vector field by hypothesis, for $X = X_3$ and $X_4 = X_2$ we arrive at

$$B(X_2,X_3) = 0.$$

Thus, M is totally geodesic. The converse is clear from (3.6).

For Lorentzian space $R_1^{(n+2)}$, we have the following corollary. □

Corollary 4.1. *Let M be a lightlike hypersurface of a Lorentzian space $R_1^{(n+2)}$ such that $Ric(\xi, X) = 0$, $\forall X \in \Gamma(TM)$, $\xi \in \Gamma(TM^{\perp})$. Then M is totally geodesic if and only if M is semi-symmetric, where Ric is the Ricci tensor of M.*

Proof. If M is a lightlike hypersurface of $R_1^{(n+2)}$. Then the screen distribution of M is a Riemannian vector bundle. From (2.21), we can see that $AX \in \Gamma(S(TM))$, $\forall X \in \Gamma(TM)$. Then, the proof follows from Theorem 4.2. □

5. Parallel and Semi-Parallel Lightlike Hypersurfaces

In this section, we give a characterization on parallel lightlike hypersurfaces of a Lorentzian manifold. In fact, it shows that there do not exist non-totally geodesic parallel lightlike hypersurfaces in a Lorentzian manifold. Moreover, we investigate the effect of semi-parallel condition on the geometry of lightlike hypersurfaces in a semi-Euclidean space.

Theorem 5.1. *Let M be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$. Then the second fundamental form of M is parallel if and only if M is totally geodesic.*

Proof. Let M be a lightlike hypersurface of a Lorentzian manifold. We suppose that the second fundamental form h is parallel. Then, from (3.3) and (2.15) we have

$$(\nabla_X h)(Y,Z) = X(B(Y,Z)N) - B(\nabla_X Y, Z)N - B(Y, \nabla_X Z)N = 0. \tag{5.1}$$

Thus, from (2.23), for $Y = \xi$, we obtain

$$-B(\nabla_X \xi, Z)N = 0.$$

By using (2.18), we have

$$B(A^*_\xi X, Z)N = 0.$$

Hence we derive $B(A^*_\xi X, Z) = 0$. Considering (2.23) we can assume that $Z \in \Gamma(S(TM))$. Thus, from (2.22), we obtain $g(A^*_\xi X, A^*_\xi Z) = 0$. Then, for $X = Z$ we get $g(A^*_\xi X, A^*_\xi X) = 0$. On the other hand, any screen distribution $S(TM)$ of a lightlike hypersurface of a Lorentzian manifold is Riemannian. Then, we have $A^*_\xi X = 0$ for any $X \in \Gamma(TM)$. Thus, proof follows from this and (2.23). The converse is clear. □

In [8], Deprez defined and studied semi-paralel hypersurface in Euclidean n space. In the rest of this section, we investigate semi-parallel lightlike hypersurface in semi-Euclidean $(n+2)$ space.

Theorem 5.2. *Let M be a semi-parallel lightlike hypersurface of semi-Euclidean $(n+2)$ space. Then either M is totally geodesic or $C(\xi, A^*_\xi U) = 0$ for any $U \in \Gamma(S(TM))$ and $\xi \in \Gamma(TM^\perp)$, where C and A^*_ξ are the second fundamental form and shape operator of the screen distribution $S(TM)$ defined in (2.19) and (2.18), respectively.*

Proof. Since M is a semi-parallel lightlike hypersurface, we have

$$h(R(X,Y)Z, W) + h(Z, R(X,Y)W) = 0.$$

By using (3.1), we obtain

$$\begin{aligned} B(X,Z)B(AY,W) &- B(Y,Z)B(AX,W) + B(X,W)B(Z,AY) \\ &- B(Y,W)B(AX,Z) = 0 \end{aligned} \tag{5.2}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Then, from (2.23) and (5.2), for $X = \xi$, we have

$$B(Y,Z)B(A\xi,W) + B(Y,W)B(A\xi,Z) = 0.$$

Thus, for $Z = W$, we obtain $B(Y,Z)B(A\xi,Z) = 0$. Now, if $B(Y,Z) = 0$, then M is totally geodesic. If $B(Y,Z) \neq 0$, then from (2.21), we have $C(\xi, A^*_\xi U) = 0$ for any $U \in \Gamma(S(TM))$.

Example 3. Consider a hypersurface M in R^4_2 given by

$$x_1 = x_2 + \sqrt{2}\sqrt{x_3{}^2 + x_4{}^2}.$$

Then, it is easy to check that M is a lightlike hypersurface. Its radical distribution is spanned by

$$\xi = \sqrt{x_3{}^2 + x_4{}^2}\frac{\partial}{\partial x_1} - \sqrt{x_3{}^2 + x_4{}^2}\frac{\partial}{\partial x_2} + \sqrt{2}x_3\frac{\partial}{\partial x_3} + \sqrt{2}x_4\frac{\partial}{\partial x_4}.$$

Then the lightlike transversal vector bundle is spanned by

$$\begin{aligned} tr(TM) \quad &= \quad span\{N = \frac{1}{4(x_3{}^2 + x_4{}^2)}(-\sqrt{x_3{}^2 + x_4{}^2}\frac{\partial}{\partial x_1} + \sqrt{x_3{}^2 + x_4{}^2}\frac{\partial}{\partial x_2} \\ &+ \quad \sqrt{2}x_3\frac{\partial}{\partial x_3} + \sqrt{2}x_4\frac{\partial}{\partial x_4})\}. \end{aligned}$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\{Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Z_2 = -x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}\}.$$

By direct computations, we obtain

$$\bar{\nabla}_X Z_1 = \bar{\nabla}_{Z_1} X = 0,\ \bar{\nabla}_\xi \xi = \sqrt{2}\xi\,, \bar{\nabla}_{Z_2}\xi = \bar{\nabla}_\xi Z_2 = \sqrt{2}Z_2,$$

and

$$\bar{\nabla}_{Z_2} Z_2 = -x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4}$$

for any $X \in \Gamma(TM)$. Then, by using Gauss formula, we obtain

$$\nabla_X Z_1 = 0,\ \nabla_{Z_2} Z_2 = -\frac{1}{2\sqrt{2}}\xi\,,\nabla_\xi Z_2 = \nabla_{Z_2}\xi = \sqrt{2}Z_2,\ \nabla_{Z_1} Z = 0$$

and

$$B(Z_2, Z_2) = -\sqrt{2}({x_3}^2 + {x_4}^2), B(Z_1, Z_2) = 0, B(Z_1, Z_1) = 0.$$

On the other hand, we have

$$\begin{aligned}
\bar{\nabla}_\xi N &= \frac{1}{2\sqrt{2}\sqrt{{x_3}^2 + {x_4}^2}}\frac{\partial}{\partial x_1} - \frac{1}{2\sqrt{2}\sqrt{{x_3}^2 + {x_4}^2}}\frac{\partial}{\partial x_2} \\
&- \frac{1}{2}\frac{x_3}{({x_3}^2 + {x_4}^2)}\frac{\partial}{\partial x_3} - \frac{1}{2}\frac{x_4}{({x_3}^2 + {x_4}^2)}\frac{\partial}{\partial x_4}, \\
\bar{\nabla}_{Z_1} N &= 0, \\
\bar{\nabla}_{Z_2} N &= -\frac{x_4}{2\sqrt{2}({x_3}^2 + {x_4}^2)}\frac{\partial}{\partial x_3} + \frac{x_3}{2\sqrt{2}({x_3}^2 + {x_4}^2)}\frac{\partial}{\partial x_4}.
\end{aligned}$$

Thus, from Weingarten formula (2.16), we have

$$A_N\xi = 0,\ A_N Z_1 = 0,\ A_N Z_2 = \frac{1}{2\sqrt{2}({x_3}^2 + {x_4}^2)} Z_2.$$

Then, from the above equations, one can show that the following equations are satisfied

$$(R(Z_1, Z_2)\,h)(Z_1, Z_1) = 0\,, (R(Z_1, Z_2)\,h)(Z_1, Z_2) = 0\,, (R(Z_1, Z_2)\,h)(Z_2, Z_2) = 0.$$

Finally, using (2.23) and definition of $(R(X, Y).h)$, we have $R(X, Y)h)(U, \xi) = 0$ for any $X, Y, U \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$. Thus, M is a non-totally geodesic semi-parallel hypersurface of R^4_2.

6. Concluding Remarks

It is known that the second fundamental forms of a lightlike hypersurface M do not depend on the vector bundles $S(TM), S(TM^\perp)$ and $tr(TM)$. Thus, the results of this paper are stable with respect to any change in the above vector bundles.

In [10], Duggal-Bejancu showed that the geometry of a Monge lightlike hypersurface of R_1^4 essentially reduces to the geometry of a leaf of its canonical screen distribution. Thus the following question naturally arises: Are there other classes of lightlike hypersurfaces whose geometry is essentially the same as that of their chosen screen distribution?

The above problem has been studied in [3], [4], [11], [12] and [18]. On the other hand it is known that the shape operator plays a key role in studying geometry of submanifolds. In [2], Atindogbe and Duggal introduced screen conformal lightlike hypersurfaces whose shape operators are conformal to shape operators of their corresponding screen distributions. Moreover, they showed that lightlike hypersurface M of a semi-Riemannian manifold$\bar{M}$ is totally geodesic, totally umbilical or minimal if and only if any leaf M' of its integrable distribution is so immersed in $\bar{M}$ as a codimension 2 non-degenerate submanifold.

In this paper, we have shown that the curvature tensor field of a screen conformal lightlike hypersurface in a semi-Euclidean space has directly related with the curvature tensor field of a leaf of its screen distribution $S(TM)$ (Theorem 3.1). Thus we have made further progress in solving above stated problem.

Finally, we note that the results of this paper are valid for a lightlike hypersurface of a flat semi-Riemannian manifold.

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