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On QR-Submanifolds of a Quaternionic Space form

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Abstract

In this paper, we investigate mixed QR-submanifolds in a quaternionic space form and pseudo umbilical QR-submanifold of a quaternionic space form under some additional condition. Finally we give a necessary condition for QR-submanifold of a quaternion Kaehler manifold such that $\dim \upsilon^{\perp} = 1$ to be a 3-quasi Sasakian Manifold.

Key Words: Quaternion Kaehler Manifold, QR-Submanifold, Pseudo-Umbilical Submanifold, Mixed Geodesic QR-Submanifold, Almost Contact 3-Structure

1. Introduction

The main purpose of this paper is to continue study of QR-submanifolds in a quaternionic space form which were started in [1]. We prove some results being QR-submanifold analogues of well known results for CR-submanifold of a complex space form.

Bejancu classified totally umbilical QR-submanifolds in a quaternion Kaehler manifold. However, it is well known that the class of pseudo umbilical submanifolds in a quaternionic space form is too wide to classify. Recently, Sato proved that any pseudo umbilical submanifolds with nonzero parallel vector field in $CP^m(c)$ is totally real submanifold. In the present paper, we have given a theorem for the pseudo umbilical QR-submanifold with nonzero parallel mean curvature vector field in a quaternionic space form similar to the obtained by Sato in the Kaehler setting. Particularly, we prove that there exist no pseudo umbilical QR-submanifold with nonzero parallel mean curvature vector field in quaternionic space form $c \neq 0$.

On the other hand, Bejancu,Kon and Yano proved that any proper mixed foliate CR-submanifold of a complex space form ($c > 0$) is complex submanifold or totally

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real submanifold. We prove that there exist no mixed foliate QR-submanifolds in a quaternionic space form ($c > 0$).

Finally, we have considered QR-submanifold of quaternion Kaehler manifold with $\dim \upsilon^{\perp} = 1$. The present author and R.Güneș, S.Keleș have shown that QR-submanifold have almost contact 3-structure in this case[4]. In this paper, we obtain a necessary condition for QR-submanifold to be a 3-quasi Sasakian manifold.

2. Preliminaries

Let $\overline{M}$ be a Riemann manifold and M be a Riemann submanifold of $\overline{M}$ with Riemann metric induced by the Riemann metric on $\overline{M}$. Denote by $TM^{\perp}$ and TM the normal and tangent bundle respectively. $\overline{\bigtriangledown}$ and $\bigtriangledown$ show the Levi-Civita connections on $\overline{M}$ and M,respectively. Moreover $\Gamma(TM)$ represents the module of differentiable sections of a vector bundle *TM.* Then the formulas of Gauss and Weingarten are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \tag{2.2}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma\left(TM^{\perp}\right)$, where $\nabla^{\perp}$ is the normal connection induced $\bigtriangledown$ on the normal bundle $TM^{\perp}$, h is the second fundamental form and A_V is the fundamental tensor of Weingarten with respect to the normal section *V.* Moreover its well known that we have

$$g(h(X, Y), V) = g\left(A_V X, Y\right). \tag{2.3}$$

Let $\overline{M}$ be a $4n$-dimensional manifold and g be a Riemann metric on $\overline{M}$. Then $\overline{M}$ is said to be a quaternion Kaehlerian manifold, if there exit a $3-$dimensional vector bundle V of type $(1,1)$with local basis of almost Hermitian structures J_1, J_2, J_3 satisfying

$$J_1 o J_2 = -J_2 o J_1 = J_3 \tag{2.4}$$

and

$$\overline{\nabla}_X J_a = \sum_{b=1}^{3} Q_{ab}(X) J_b, a = 1, 2, 3 \tag{2.5}$$

for all vector fields X tangent to $\overline{M}$,where Q_{ab} are certain 1-forms locally defined on $\overline{M}$ such that $Q_{ab} + Q_{ba} = 0$

Let $\overline{M}$ be quaternion Kaehler manifold and M be a real submanifold of $\overline{M}$. Then, M is said QR-submanifold if there exists a vector subbundle ν of the normal bundle such that we have

$$J_a(\nu_x) = \nu_x \tag{2.6}$$

and

$$J_a(\nu_x^{\perp}) \subset T_M\,(x) \tag{2.7}$$

for $x \in M$ and $a = 1, 2, 3$, where $\nu^{\perp}$ is the complementary orthogonal bundle to ν in $TM^{\perp}$[1]. Let M be a QR-submanifold of $\overline{M}$. Set $D_{ax} = J_a(\nu_x^{\perp})$. We consider $D_{1x} \oplus D_{2x} \oplus D_{3x} = D_x^{\perp}$ and $3s-$ dimensional distribution $D^{\perp} : x \to D_x^{\perp}$ globally defined on M,where $s = \dim \nu_x^{\perp}$. Also we have, for each $x \in M$

$$J_a(D_{ax}) = \nu_x^{\perp}, J_a(D_{bx}) = D_{cx} \tag{2.8}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. We denote by D the complementary orthogonal distribution to $D^{\perp}$ in TM. Then D is invariant with respect to the action of J_a i.e. we have

$$J_a(D_x) = D_x \tag{2.9}$$

for any $x \in M$. D is called quaternion distribution.

Let M be a QR-submanifold of a quaternion Kaehler $\overline{M}$. Denote by P the projection morphism of TM to the quaternion distribution D and choose a local field of orthonormal frames $\{v_1, ..., v_s\}$ on the vector subbundle $\nu^{\perp}$ in $TM^{\perp}$. Then on the distribution $D^{\perp}$, we have the local field of orthonormal frames

$$\{E_{11},...,E_{1s},E_{21},...,E_{2s},E_{31},...,E_{3s}\} \tag{2.10}$$

where $E_{ai}=J_a v_i$, $a=1,2,3$ and $i=1,...,s$. Thus any vector field Y tangent to M can be written locally as follows

$$Y=PY+\sum_{b=1}^{3}\sum_{i=1}^{s}W_{bi}(Y)E_{bi} \tag{2.11}$$

where W_{bi} are 1-forms locally defined on M by

$$W_{bi}(Y)=g(Y,E_{bi}). \tag{2.12}$$

Applying J_a to (2.11) and taking account of (2.4) we have

$$\begin{aligned} J_aY &= J_aPY+\sum_{i=1}^{s}\{W_{bi}(Y)E_{ci}-W_{ci}(Y)E_{bi} \\ &\quad -W_{ai}(Y)v_i \end{aligned} \tag{2.13}$$

where (a,b,c) is a cyclic permutation of $(1,2,3)$. For $Y\in\Gamma(TM)$ we can decompose as follows

$$J_aY=\phi_aY+F_aY, a=1,2,3 \tag{2.14}$$

where ϕ_aY and F_aY the tangential and normal parts of J_aY, respectively. Similar way we get

$$J_aV=t_aV+f_aV. \tag{2.15}$$

We note that a QR-submanifold is called mixed geodesic if $h(X,Y)=0$ for $X\in\Gamma(D)$ and $Y\in\Gamma\left(D^{\perp}\right)$ and M is called mixed foliate if the distribution D is integrable and M is mixed geodesic [2].

Now,we state the following well known result for later use.

Theorem 2.1 *([1]) Let M be a QR-submanifold of a quaternion Kaehlerian manifold $\overline{M}$. Then the following assertions are equivalent with each other*

(i)the second fundamental form of M satisfies

$$h(X, J_aY) = h(Y, J_aX)$$

for any $X, Y \in \Gamma(D)$,$a = 1, 2, 3$.

(ii) M is $D-$ geodesic

(iii)the quaternion distribution D is involutive.

A quaternionic space form is a connected quaternion Kaehler manifold of constant quaternionic sectional curvature and its denoted by $\overline{M}(c)$. The curvature tensor of $\overline{M}(c)$ is given by [7]

$$\begin{aligned} \overline{R}(X,Y)Z = \; & \tfrac{c}{4}\{g(Z,Y)X - g(X,Z)Y \\ & + \sum_{a=1}^{3} g(Z, J_aY)J_aX - g(Z, J_aX)J_aY \\ & +2g(X, J_aY)J_aZ\} \end{aligned} \tag{2.16}$$

for any $X, Y, Z \in \Gamma\left(T\overline{M}\right)$.

A normal vector field V is said to be parallel if $\nabla_X V = 0$ for all $X \in \Gamma(TM)$. Let $H = \frac{1}{n} traceh$ be the mean curvature vector of M in $\overline{M}$. If second fundamental form h is of the form $g(h(X,Y), H) = g(X,Y)g(H,H)$ for $X, Y \in \Gamma(TM)$, then M is said to be pseudo umbilical or equivalently $A_H = \|H\|^2 I$.

For the second fundamental form h, the covariant derivation $(\nabla_X h)(Y,Z)$ is given by

$$(\nabla_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.17}$$

for any $X, Y, Z \in \Gamma(TM)$. On the other hand, for the submanifold M the equations of Gauss and Codazzi are respectively

$$\begin{aligned} R(X,Y,Z,W) \;&=\; \overline{R}(X,Y,Z,W) + g(h(X,W), h(Y,Z)) \\ &\quad -g(h(X,Z), h(Y,W)) \end{aligned} \tag{2.18}$$

$$\left(\overline{R}(X,Y)Z\right)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \tag{2.19}$$

for any $X, Y, Z, W \in \Gamma(TM)$ [7].

3. On QR-Submanifolds of a Quaternionic Space Form

We shall give some lemmas for later use.

Lemma 3.1 *Let $\overline{M}$ be a quaternion Kaehlerian manifold and M be a QR-submanifold of $\overline{M}$. Then M is mixed geodesic if and only if*

$$A_V X \in \Gamma(D) \tag{3.1}$$

for any $X \in \Gamma(D)$ and $V \in \Gamma\left(TM^{\perp}\right)$.

Proof. By the definition of mixed geodesic QR-submanifold and from the equation (2.3) we have the assertion of the lemma. □

Lemma 3.2 *Let $\overline{M}$ be a quaternion Kaehlerian manifold and M be a mixed geodesic QR-submanifold of $\overline{M}$. Then*

$$A_{J_a W_i} X = J_a A_{W_i} X \tag{3.2}$$

and

$$\nabla^{\perp}_X W_i \in \Gamma(\nu) \tag{3.3}$$

$X \in \Gamma(D)$ and $W_i \in \Gamma(\nu)$.

Proof. From (2.5) we obtain

$$\begin{aligned}
\overline{\nabla}_X J_a W_i &= \left(\overline{\nabla}_X J_a\right) W_i + J_a \overline{\nabla}_X W_i \\
&= Q_{ab}(X) J_b W_i + Q_{ac}(X) J_c W_i \\
&\quad + J_a \overline{\nabla}_X W_i.
\end{aligned}$$

By using(2.1) and (2.2) we have

$$\begin{aligned}
-A_{J_a W_i} X + \nabla^{\perp}_X J_a W_i &= Q_{ab}(X) J_b W_i + Q_{ac}(X) J_c W_i \\
&\quad - J_a A_{W_i} X + J_a \nabla^{\perp}_X W_i
\end{aligned}$$

since M is mixed geodesic, from (3.1) we derive

$$A_{J_a W_i} X = J_a A_{W_i} X$$

and

$$\nabla^{\perp}_X J_a W_i - Q_{ab}(X) J_b W_i - Q_{ac}(X) J_c W_i = J_a \nabla^{\perp}_X W_i.$$

The left hand side of this equation belongs to $TM^{\perp}$, thus $\nabla^{\perp}_X W_i \in \Gamma(\nu)$.

□

Lemma 3.3 *Let M be a foliate QR-submanifold of quaternion Kaehler manifold. Then we have the following expression;*

$$g(A_{V_i} X, J_a Y) = g(A_{V_i} J_a X, Y) \tag{3.4}$$

for any $X, Y \in \Gamma(D)$, $V_i \in \Gamma(\nu^{\perp})$

Proof. From (2.3) we have

$$g(A_{V_i} X, J_a Y) = g(h(X, J_a Y), V_i)$$

since D is integrable, from theorem2.1 we get

$$g(A_{V_i} X, J_a Y) = g(h(J_a X, Y), V_i)$$

Thus we have $g(A_{V_i} X, J_a Y) = g(A_{V_i} J_a X, Y)$.

□

Lemma 3.4 *Let M be a mixed geodesic QR-submanifold of quaternion Kaehler manifold. Then we have the following expression;*

$$\nabla_Y E_{ai} = Q_{ab}(Y) E_{bi} + Q_{ac}(Y) E_{ci} - J_a A_{V_i} Y + B_a \nabla_Y V_i \tag{3.5}$$

for any $Y \in \Gamma(D)$, $E_{ai} \in \Gamma(D^{\perp})$.

Proof. By using (2.5), (2.1) and (2.2) we have

$$\begin{aligned}
\overline{\nabla}_X E_{ai} &= \left(\overline{\nabla}_X J_a\right) V_i + J_a \overline{\nabla}_X V_i \\
&= Q_{ab}(X) J_b V_i + Q_{ac}(X) J_c V_i \\
&\quad + J_a \overline{\nabla}_X V_i. \\
\nabla_X E_{ai} + h(X, E_{ai}) &= Q_{ab}(X) E_{bi} + Q_{ac}(X) E_{ci} \\
&\quad + J_a(-A_{V_i} X + \nabla_X^{\perp} V_i) \\
&= Q_{ab}(X) E_{bi} + Q_{ac}(X) E_{ci} \\
&\quad - J_a A_{V_i} X + B_a \nabla_X^{\perp} V_i \\
&\quad + C_a \nabla_X^{\perp} V_i
\end{aligned}$$

Taking acount of that M is mixed geodesic we obtain

$$\begin{aligned}
\nabla_X E_{ai} &= Q_{ab}(X) E_{bi} + Q_{ac}(X) E_{ci} \\
&\quad - J_a A_{V_i} X + B_a \nabla_X^{\perp} V_i.
\end{aligned}$$

□

Theorem 3.5 *There exist no mixed foliate QR-submanifold of quaternionic space form with $c > 0$.*

Proof. We suppose that M is mixed foliate QR-submanifold of quaternionic space form with $c > 0$.First,from (2.16) and (2.19) we get

$$\left(\nabla_X h\right)(Y, E_{ai}) - \left(\nabla_Y h\right)(X, E_{ai}) = -\frac{c}{2} g(X, J_a Y) V_i$$

for any $X, Y \in \Gamma(D)$ and $E_{ai} \in \Gamma\left(D^{\perp}\right)$. On the other hand, since M is mixed foliate we derive

$$h(X, \nabla_Y E_{ai}) - h(Y, \nabla_X E_{ai}) = -\frac{c}{2} g(X, J_a Y) V_i,$$

from (3.5) we have

$$-h(X, J_a A_{V_i} Y) + h(Y, J_a A_{V_i} X) = -\frac{c}{2} g(X, J_a Y) V_i.$$

Since M is mixed geodesic, $A_{V_i} Y \in \Gamma(D)$,from Theorem 1 we derive

$$-h(J_a X, A_{V_i} Y) + h(J_a Y, A_{V_i} X) = -\frac{c}{2} g(X, J_a Y) V_i.$$

Thus for $X = J_a Y$ we derive

$$h(Y, A_{V_i} Y) + h(J_a Y, A_{V_i} J_a Y) = -\frac{c}{2} g(Y, Y) V_i$$

or

$$g\left(h(Y, A_{V_i} Y), V_i\right) + g\left(h(J_a Y, A_{V_i} J_a Y), V_i\right) = -\frac{c}{2} g(Y, Y) g\left(V_i, V_i\right)$$

by using (2.3) we obtain

$$g\left(A_{V_i} Y, A_{V_i} Y\right) + g\left(A_{V_i} J_a Y, A_{V_i} J_a Y\right) = -\frac{c}{2} g(Y, Y) g\left(V_i, V_i\right)$$

from (3.1) and (3.4) we get

$$\begin{aligned} g\left(A_{V_i} Y, A_{V_i} Y\right) + g\left(A_{V_i} Y, J_a A_{V_i} J_a Y\right) &= -\frac{c}{2} g(Y, Y) g\left(V_i, V_i\right) \\ g\left(A_{V_i} Y, A_{V_i} Y\right) + g\left(A_{V_i} Y, A_{V_i} Y\right) &= -\frac{c}{2} g(Y, Y) g\left(V_i, V_i\right) \\ 2g\left(A_{V_i} Y, A_{V_i} Y\right) &= -\frac{c}{2} g(Y, Y) g\left(V_i, V_i\right) \end{aligned}$$

thus we have

$$0 \leq 2g\left(A_{V_i} Y, A_{V_i} Y\right) = -\frac{c}{2} g(Y, Y) g\left(V_i, V_i\right)$$

which proves assertion.

□

Theorem 3.6 *There exist no pseudo umbilical QR-submanifold of a quaternionic space form $\overline{M}(c), c \neq 0$ with nonzero parallel mean curvature vector field.*

Proof. We suppose that M be a pseudo umbilical submanifold of $\overline{M}(c), c \neq 0$ with nonzero parallel mean curvature vector field. Then we have

$$\overline{\nabla}_X H = -A_H X = -\|H\|^2 X, \forall X \in \Gamma(TM)$$

where $\|H\|$ is a constant . Therefore we have

$$\overline{R}(X,Y)H = 0.$$

On the other hand, from (2.16) we get

$$g\left(\overline{R}(X,Y)H, J_1 H\right) = \frac{c}{2} g(X, J_1 Y) g(H,H)$$

for any $X, Y \in \Gamma(D)$. Since D is nondegenerate and $g(H,H) \neq 0$ we have $c = 0$

□

4. A Theorem on QR-Submanifolds in Quaternion Kaehler Manifolds with $\dim \nu^{\perp} = 1$

Let N be $(4m+3)-$dimensional differentiable manifold and (ϕ_a, ξ_a, η_a) be three almost contact structures on N. i.e. We have

$$\phi_a^2 X = -X + \eta_a(X)\xi_a, \phi_a \xi_a = 0 \tag{4.1}$$

$$\eta_a(\xi_a) = 1, \eta_a o \phi_a = 0 \tag{4.2}$$

where X tangent to N. Suppose that almost contact structures satisfy the following conditions

$$\eta_a(\xi_b) = 0, a \neq b, \phi_a(\xi_b) = -\phi_b(\xi_a) = \xi_c \tag{4.3}$$

$$\eta_a o \phi_b = -\eta_b o \phi_a = \eta_c \tag{4.4}$$

$$(\phi_a o \phi_b)(X) - \xi_a(\eta_b(X)) = (\phi_b o \phi_a)(X) - \xi_b(\eta_a(X)) = \phi_c X \tag{4.5}$$

for any cyclic permutation (a, b, c) of $(1, 2, 3)$. Then,we say that N is endowed with an almost contact $3-$structure [5].If N is a Riemannian manifold, then we can choose a Riemann metric g on M such that we have

$$g(\phi_a X, \phi_a Y) = g(X, Y) - \eta_a(X)\eta_a(Y) \tag{4.6}$$

$$\eta_a(X) = g(X, \xi_a) \tag{4.7}$$

for any $X, Y \in \Gamma(TN)$. In this case we say that $(\phi_a, \xi_a, \eta_a), a = 1, 2, 3.$ define almost contact metric structure (see,[5]). Taking account of (4.1) and (4.6), we obtain

$$g(\phi_a X, Y) + g(X, \phi_a Y) = 0. \tag{4.8}$$

Definition 4.1 *An almost contact* $3-$*structure* (ϕ_a, ξ_a, η_a) *is*

a) a $3-$*cosymplectic structure if*

$$(\nabla_X \phi_a) Y = 0, (\nabla_X \eta_a) Y = 0 \tag{4.9}$$

b) a $3-$*Sasakian structure if*

$$(\nabla_X \phi_a) Y = \eta_a(Y)X - g(X, Y)\xi_a \tag{4.10}$$

c) a quasi-$3-$*Sasakian if*

$$g((\nabla_X \phi_a) Y, Z) + g((\nabla_Y \phi_a) Z, X) + g((\nabla_Z \phi_a) X, Y) = 0 \tag{4.11}$$

where ∇ *denotes the Levi-Civita connection and* X, Y, Z *are arbitrary vector fields on* N.

Now, Let M be a QR-submanifold of quaternion Kaehler manifold $\overline{M}$ such that the dimension $\nu^\perp$ is equal to one. In this case $\nu^\perp$ is generated by unit vector field, say N. Let $-J_a(N) = \xi_a, a = 1, 2, 3.$ and hence the distributions D_a are generated by the vector fields ξ_a.Since $\nu^\perp$ is generated by unit vector field, we have

$$J_a Y = \phi_a Y + \eta_a(Y)N \tag{4.12}$$

for any $Y \in \Gamma(TM)$, where $\eta_a(Y) = g(Y, \xi_a)$.

In this section we will make use of the following proposition whose proof was given in [4].

From now on we will denote by M a QR-submanifold with $\dim \nu^\perp = 1$.

Proposition 4.2 *Let $\overline{M}$ be a quaternion Kaehler manifold and M be QR-submanifold of $\overline{M}$. Then M is a manifold with almost contact 3-structure. i.e. tensor field ϕ_a of type (1,1),$1-$form η_a and ξ_a satisfy (4.1)-(4.7)*

Let M be a QR-submanifold of quaternion Kaehler manifold $\overline{M}$. Then by using (2.1), (2.2),(2.3),(2.14) and (4.12) in (2.5) and taking the tangent parts we obtain

$$\begin{aligned} g\left(\left(\nabla_X \phi_a\right) Y\right), Z) &= \eta_a(Y)\alpha(X,Z) - \alpha(X,Y)\eta_a\left(Z\right) \\ &+\{\alpha\left(X,\xi_c\right) + \eta_b\left(\nabla_X \xi_a\right)\} g\left(\phi_b Y, Z\right) \\ &+\{-\alpha\left(X,\xi_b\right) + \eta_c\left(\nabla_X \xi_a\right)\} g\left(\phi_c Y, Z\right)) \end{aligned} \tag{4.13}$$

for any $X, Y, Z \in \Gamma\left(TM\right)$

Theorem 4.3 *Let $\overline{M}$ be a quaternion Kaehler manifold and M be QR-submanifold of $\overline{M}$. If $h(X,\xi_a), a = 1,2,3$ have no components in $\nu^{\perp}$ and $D_a, a = 1,2,3$ are parallel in M. Then M is a manifold with quasi Sasakian 3-structure.*

Proof. From (4.13) we have

$$\begin{aligned} g\left(\left(\nabla_X \phi_a\right) Y\right), Z) &= \eta_a(Y)\alpha(X,Z) - \alpha(X,Y)\eta_a\left(Z\right) \\ &+\{\alpha\left(X,\xi_c\right) + \eta_b\left(\nabla_X \xi_a\right)\} g\left(\phi_b Y, Z\right) \\ &+\{-\alpha\left(X,\xi_b\right) + \eta_c\left(\nabla_X \xi_a\right)\} g\left(\phi_c Y, Z\right)), \end{aligned} \tag{4.14}$$

$$\begin{aligned} g\left(\left(\nabla_Y \phi_a\right) Z\right), X) &= \eta_a(Z)\alpha(Y,X) - \alpha(Y,Z)\eta_a\left(X\right) \\ &+\{\alpha\left(Y,\xi_c\right) + \eta_b\left(\nabla_Y \xi_a\right)\} g\left(\phi_b Z, X\right) \\ &\{-\alpha\left(Y,\xi_b\right) + \eta_c\left(\nabla_Y \xi_a\right)\} g\left(\phi_c Z, X\right)) \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} g\left(\left(\nabla_Z \phi_a\right) X\right), Y) &= \eta_a(X)\alpha(Z,Y) - \alpha(Z,X)\eta_a\left(Y\right) \\ &-\{\alpha\left(Z,\xi_c\right) + \eta_b\left(\nabla_Z \xi_a\right)\} g\left(\phi_b X, Y\right) \\ &+\{-\alpha\left(Z,\xi_b\right) + \eta_c\left(\nabla_Z \xi_a\right)\} g\left(\phi_c X, Y\right)), \end{aligned} \tag{4.16}$$

for any $X, Y, Z \in \Gamma(TM)$. Thus using (4.14),(4.15) and (4.16) we obtain

$$\begin{aligned} & g\left(\left(\nabla_X \phi_a\right) Y\right), Z) + g\left(\left(\nabla_Y \phi_a\right) Z\right), X) \\ +g\left(\left(\nabla_Z \phi_a\right) X\right), Y) \quad = \quad & \{\alpha\left(X, \xi_c\right) + \eta_b\left(\nabla_X \xi_a\right)\} g\left(\phi_b Y, Z\right) \\ & +\{-\alpha\left(X, \xi_b\right) + \eta_c\left(\nabla_X \xi_a\right)\} g\left(\phi_c Y, Z\right)) \\ & +\{\alpha\left(Y, \xi_c\right) + \eta_b\left(\nabla_Y \xi_a\right)\} g\left(\phi_b Z, X\right) \\ & +\{-\alpha\left(Y, \xi_b\right) + \eta_c\left(\nabla_Y \xi_a\right)\} g\left(\phi_c Z, X\right)) \\ & +\{\alpha\left(Z, \xi_c\right) + \eta_b\left(\nabla_Z \xi_a\right)\} g\left(\phi_b X, Y\right) \\ & +\{-\alpha\left(Z, \xi_b\right) + \eta_c\left(\nabla_Z \xi_a\right)\} g\left(\phi_c X, Y\right)) \end{aligned}$$

Hence if $D_a, a = 1, 2, 3$ are parallel and $\alpha(X, \xi_a) = 0$, then M is a manifold with 3-quasi Sasakian structure.

□

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