Robust Stabilizing *PID* Controllers for Multiple Time Delay Systems with Parametric Uncertainty

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Abstract: This paper presents a methodology to compute all stabilizing robust PI and PID controllers for multiple time delay systems with parametric uncertainty structure. The method is based on the computation of the stability regions in a parameter space of the PI and PID controllers. Analytical expressions are derived to construct the boundaries of the stability region. All values of the controller parameters in the proposed stability region guarantee the robust stability of multiple time delay systems. A case study illustrates the effectiveness of the proposed method.

Keywords: Parameter space, Multiple time delay, Parameter uncertainty, Stabilizing controllers.

1. INTRODUCTION

Time delay is unavoidable in practice with real control systems due to measurement lags, analysis times or computation lags as well as a number of other factors (Fridman et al., 2010). Control system designers may sometimes neglect relatively small delays in which the systems still satisfy design requirements. But time delay may cause instability in real applications, and its effects cannot be underestimated. Therefore, the stability problems of the time delay systems have been main topic of study for many researchers over the last few decades (Debeljković, 2010). Several studies on different kind of time delay structure have been proposed in literature due to its importance in control applications. It is reported that the multiple time delay structure is one of the important kind of time delay in the literature (Sipahi and Olgac, 2006). Thus, the results on multiple time delay systems will contribute researches in this direction.

Although the stability problems of the control systems with only one single delay have been studied extensively, limited research has been done analyzing the stability of multiple time delay systems, which are more complex to solve (Sipahi and Olgac, 2006). Multiple time delay system exhibit hyper-chaos and have more complicated dynamics than in single delay systems (Debeljković, 2010). In recent studies, these systems have become a topic of interest due to their potential applications in various fields. Many such studies on multiple time delay systems can be found in (Lua et al., 2005; Wang et al., 2009; He et al., 2011; Ma et al., 2010; Fridman and Shaked, 2002) and the references therein.

On the other hand, parameter uncertainty, which also disturb the stability of the control process, is one of the main topics for real systems. These uncertainties may be due to additive unknown internal or external noises, environmental influences, external disturbances, and parameter perturbations (Ma et al., 2010). Not only the problem of controlling time delay systems has become an important subject over the last few decades, but also controlling time delay systems in the presence of uncertainty has become a challenging task for researchers (Hien and Phat, 2009). Consequently, investigation of the methods for uncertain multiple time delay systems' stability problems will be important.

Several controller tuning methods have been proposed for stability problems of control systems. Controller tuning for a control system is a process of obtaining the controller parameters required to meet given performance specifications. The most common controllers in industrial applications are the proportional (*P*), proportional-integral (*PI*), proportional derivative (*PD*) and proportional integral derivative (*PID*) controllers due to their simplicity and reliability (Xue et al., 2007). There have been a significant amount of researches on *PI*, *PD* and *PID* controllers in the literature. Stability regions of these controller parameters have been investigated during the last decade and many important results have been reported on the computation of all stabilizing *P*, *PI* and *PID* controllers after the publication of a study by Ho et al. (Ho et al., 1996; Ho et al., 1997). This approach has shown that for a fixed proportional gain, the set of stabilizing integral and derivative gains lie in a convex set. Stabilizing controllers have been the focus of several studies over the last decade due to the importance of the stability region in controllers. Several such studies have been previously discussed for the integer order and fractional order systems with time delays in (Tan et al., 2006; Hamamci, 2007; Hamamci, 2008; Tan, 2003) and the references therein.

Since the stability of multiple time delay systems is challenging research area, stability analysis of multiple time-delay systems with applications are extensively studied (Delice, 2011) and these systems become one of the active areas of control research. Thus, multiple time delay systems with parametric uncertainty warrant attention regarding their stabilizing controllers due to their importance in various real processes. The novel contribution of this paper lies in computing all stabilizing *PI* and *PID* controllers in (k_p, k_i) and (k_p, k_i, k_d) parameter space respectively that guarantee the robust stability of the multiple time delay systems with parametric uncertainty under all parameter perturbations.

In this paper, the expressions for the *PI* and *PID* controller parameters k_p, k_i and k_d are derived. The complex root boundary of a system in the parameter space is obtained using the expressions for the parameters k_p, k_i and k_d. The real root boundary and the infinite root boundary of a system are calculated using the closed loop characteristic polynomial of an uncertain multiple time delay system that is controlled with *PI* or *PID* controller. Then, all stabilizing parameters of the robust *PI* and *PID* controllers are calculated for the system under all parameter perturbations. The proposed methodology can also be extended to the various types of the multiple time delay system that is separated into even and odd parts. One can obtain expressions for the controller parameters k_p, k_i and k_d using even and odd parts of the plant. Then, stability boundary can be computed for the desired plant.

Rest of the paper is organized as follows: The problem statement and preliminaries are introduced in Section II. In Section III, a method is proposed for calculating robust stabilizing controllers. A case study is presented in Section IV. Finally, Section V presents the conclusions.

2. PROBLEM STATEMENT AND PRELIMINARIES

The state space representation is a mathematical model of a physical system as a set of input, output and state variables related to first-order differential equations. The task of modeling state space representation of a system is to obtain the elements of the matrices and to write the system model in the following form (He et al., 2011),

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

The matrices A and B are properties of the system that are determined by the system's structure and elements. The output equation matrices C and D are determined by the particular choice of output variables. A linear continuous-time system with multiple time delays can be represented as (He et al., 2011; He et al., 2006),

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{j=1}^{m} A_j x(t-\tau_j) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{2}$$

where A_0 and A_j $(j = 1, 2, \cdots m)$ are $(n \times n)$ dimensional state matrices, B is the input matrix of $(n \times p)$ dimensional, C is the output matrix of $(q \times n)$ dimensional, D is the direct transmission matrix of $(q \times p)$ dimensional, $x(t) \in R^n$ is the system state vector, $u(t) \in R^p$ is the system input vector, $y(t) \in R^q$ is the system output vector and $\tau_j > 0$ $(j = 1, 2, \cdots m)$ are constant state delays in the system. Therefore the transfer function of this system can be obtained as follows,

$$G(s) = \frac{N(s)}{D(s)} = C\left(sI - A_0 - \sum_{j=1}^{m} A_j e^{\tau_j s} \right)^{-1} B + D \tag{3}$$

The stability analysis of time delay systems with or without uncertainties has been an active research area for the past decades (Cao et al., 2003). Time delays and uncertainties may frequently cause instability in real control applications. Consequently, the stability analysis of linear continuous-time multiple state delayed uncertain systems has attracted the attention of the researchers. The compensator design in classical control engineering is based on plants with fixed parameters. In the real world, however, most practical system models are not precisely known, meaning that systems contain uncertainties (Cao et al., 2003). Much recent work on systems with uncertain parameters has been based on Kharitonov's findings (Kharitonov, 1979) on the stability of interval polynomials. Using the Kharitonov theorem, there have been many developments in the field of parametric robust control related to the stability and performance analysis of uncertain control systems, which are represented as interval plants (Bhattacharyya et al., 1995).

In classical control, the transfer function of a system with parametric uncertainty can be defined as follows,

$$G(s) = \frac{N(s)}{D(s)} = \frac{q_m s^m + q_{m-1} s^{m-1} + \cdots + q_0}{p_n s^n + p_{n-1} s^{n-1} + \cdots + p_0} \tag{4}$$

where $q_i \in [\underline{q_i}, \overline{q_i}]$ $, i = 1, 2, \ldots m$ and $p_j \in [\underline{p_j}, \overline{p_j}]$, $j = 1, 2, \ldots n$. The parameters $\underline{q_i}$ and $\underline{p_j}$ represent the lower limits, and $\overline{q_i}$ and $\overline{p_j}$ represents the upper limits of the uncertain parameters q_i and p_j, respectively. Similarly, the state space representation of a system may also be defined at a certain interval due to the tolerance values of the system's parameters. In that case, the $(n \times n)$ dimensional state matrix of a system with parametric uncertainty can be given as follows,

$$A_j = \begin{bmatrix} [\underline{a_{j(1,1)}} & \overline{a_{j(1,1)}}] & \cdots & [\underline{a_{j(1,n)}} & \overline{a_{j(1,n)}}] \\ [\underline{a_{j(2,1)}} & \overline{a_{j(2,1)}}] & \cdots & [\underline{a_{j(2,n)}} & \overline{a_{j(2,n)}}] \\ \vdots & & & \\ \vdots & & & \\ [\underline{a_{j(n,1)}} & \overline{a_{j(n,1)}}] & \cdots & [\underline{a_{j(n,n)}} & \overline{a_{j(n,n)}}] \end{bmatrix} \tag{5}$$

$(j = 0, 1, 2, \cdots m)$

where $\underline{a_{j(1,1)}}$, $\underline{a_{j(1,2)}}$, $\cdots$ $\underline{a_{j(n,n)}}$ represent the lower limits of the parameters of the coefficient matrices while $\overline{a_{j(1,1)}}$, $\overline{a_{j(1,2)}}$, $\cdots$ $\overline{a_{j(n,n)}}$ represent the upper limits. The transfer function of the state space representation of multiple time delay systems with parametric uncertainty is required to compute all stabilizing controllers for these types of systems. Interval representation of the transfer function is obtained by substituting (5) in (3) for this case.

3. COMPUTATION OF ROBUST STABILIZING PID CONTROLLERS

Stabilization problem are widely studied due to importance in control applications (Sarjaš et al., 2011) and robust controller design (Chowdhury et al., 2011). Thus, stabilizing controllers for different kind of systems are promising subject. New studies fill some gaps in stabilizing controller researches. Such as, an approach of stabilization and control of time invariant systems of arbitrary order, which include several time delays in the form of $G(s)=\sum_{i=1}^{N} G_i(s)e^{-\tau_i s}$, was recently proposed by (Karoui et al. 2013). However, the robust stabilizing controllers design for multiple time delay systems with parametric uncertainty still needs investigation.

In this section, we extended the stabilizing controller design problem, which is discussed in (Tan et al., 2006; Hamamci, 2007; Hamamci, 2008; Tan, 2003) and the references therein, to multiple time delay systems with parametric uncertainty. The proposed stabilizing controllers satisfy the robust performance of the uncertain system. Consider the closed loop control system in Fig. 1. Let the transfer function of the plant be multiple time delay system with parametric uncertainty and let the controller be *PID* type. The stability of this system can be investigated for all stabilizing values of *PID* controllers.

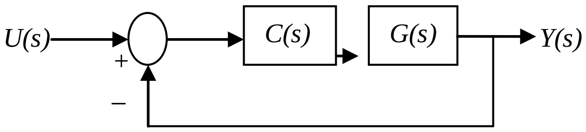

Fig. 1. Negative unity feedback control system.

Consider the plant in Fig. 1 $G(s)=N(s)/D(s)$ and the *PID* controller $C(s)$ of the following form,

$$C(s)=k_p+\frac{k_i}{s}+k_d s \tag{6}$$

The characteristic equation of the system is

$$\Delta(s)=\delta(s,k_p,k_i,k_d)=1+C(s)G(s)=0 \tag{7}$$

The closed loop characteristic polynomial $\Delta(s)$ of the system in Fig. 1 can be obtained as,

$$\Delta(s)=sD(s)+(k_d s^2+k_p s+k_i)N(s) \tag{8}$$

The closed loop system is said to be bounded input bounded output stable if the quasipolynomial $\delta(s,k_p,k_i,k_d)$ has no roots in the open right half of the *s*-plane. In other words, all roots of the quasipolynomial $\delta(s,k_p,k_i,k_d)$ lie in the closed left half of the *s*-plane for the stability domain in the parameter space of k_p, k_i and k_d. Therefore, it is important to determine the stability domain for the controller design to control the multiple time delay system with parametric uncertainty. The stability boundary in the parameter space of the *PI* or *PID* controller can be determined by real root, complex root and infinite root boundaries of the system. Then, the stability region can be easily computed to satisfy stability conditions using these stability boundaries of the system.

The stability boundaries can be defined as (Hamamci, 2007; Tan, 2003):

Definition 1: Real Root Boundary (RRB): A real root crosses over the imaginary axis at $s=0$. Consequently, the real root boundary is obtained by substituting $s=0$ in the characteristic polynomial of a closed loop system.

Definition 2: Complex Root Boundary (CRB): A pair of complex roots crosses over the imaginary axis at $s=j\omega$. Therefore, the real and imaginary parts of characteristic polynomial of the closed loop system simultaneously become zero.

Definition 3: Infinite Root Boundary (IRB): A real root crosses over the imaginary axis at $s=\infty$. Thus, the infinite root boundary can be characterized by equating the higher order term of the characteristic polynomial to zero.

The transfer function and the characteristic polynomial of a closed loop system will be calculated to obtain the RRB, CRB and IRB. Consider the transfer functions of the controller and plant. Substituting $s=j\omega$, the numerator and denominator of the $C(s)$ and $G(s)$ can be written as follows (Ho et al., 1996, Tan, 2003),

$$\begin{aligned} C(j\omega)&=\frac{-\omega^2 k_d+j\omega k_p+k_i}{j\omega}\\ G(j\omega)&=\frac{N_e(-\omega^2)+j\omega N_o(-\omega^2)}{D_e(-\omega^2)+j\omega D_o(-\omega^2)} \end{aligned} \tag{9}$$

where $N_e(-\omega^2)$ and $N_o(-\omega^2)$ are the even and odd parts of the $N(s)$, similarly $D_e(-\omega^2)$ and $D_o(-\omega^2)$ are the even and odd parts of the $D(s)$, respectively. One can easily decompose the transfer function of the system to its real and imaginary parts. Similarly, the characteristic polynomial of the closed loop system $\Delta(j\omega)=1+C(j\omega)G(j\omega)=0$ can be separated into real and imaginary parts as $\Delta(s)=R_\Delta+jI_\Delta=0$. For simplicity $(-\omega^2)$ will be dropped in the following equations. Thus the following equations can be written for real and imaginary parts of the characteristic equation as,

$$R_\Delta(\omega)=N_e k_i-\omega^2 N_o k_p-\omega^2 N_e k_d-\omega^2 D_o=0 \tag{10}$$

$$jI_\Delta(\omega)=j(\omega N_e k_p+\omega N_o k_i-\omega^3 N_o k_d+\omega D_e)=0 \tag{11}$$

Solving (10) and (11), one can obtain the expressions for k_p and k_d as follows,

$$k_p=\frac{-\omega^2 N_o D_o-N_e D_e}{N_e^2+\omega^2 N_o^2} \tag{12}$$

$$k_d = \frac{(N_e^2 + \omega^2 N_o^2)k_i + \omega^2 (N_o D_e - N_e D_o)}{\omega^2 (N_e^2 + \omega^2 N_o^2)} \tag{13}$$

Equations (12) and (13) are depended to parameter k_i. Namely, a stabilizing region in (k_p, k_d) parameter space can be obtained for different values of k_i. Similarly, the expressions for k_i can also be obtained as,

$$k_i = \frac{\omega^2 (N_e^2 + \omega^2 N_o^2)k_d + \omega^2 (N_e D_o - N_o D_e)}{N_e^2 + \omega^2 N_o^2} \tag{14}$$

In this case, the (14) depends to the parameter k_d. Thus, a stabilizing region in (k_p, k_i) parameter space can be obtained for different values of k_d. Consequently, the CRB can be computed in the parameter space of the quasipolynomial $\delta(s, k_p, k_i, k_d)$ using (12)-(14).

In this section, we will discuss a second order state space system. The method presented in this paper, can be applied to the higher order systems in a similar manner. Now, consider the second order state space system in the form of (2) as follows,

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{15}$$

where,

$$A_0 = \begin{bmatrix} a_{0(1,1)} & a_{0(1,2)} \\ a_{0(2,1)} & a_{0(2,2)} \end{bmatrix},\ A_1 = \begin{bmatrix} a_{1(1,1)} & a_{1(1,2)} \\ a_{1(2,1)} & a_{1(2,2)} \end{bmatrix},$$
$$A_2 = \begin{bmatrix} a_{2(1,1)} & a_{2(1,2)} \\ a_{2(2,1)} & a_{2(2,2)} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix} \text{ and } D = 0$$

The transfer function of the system can be obtained in controllable and observable form using the values of the *A*, *B*, *C* and *D* matrices. Using the given matrices in (15), the transfer function of the system in the form of (3) can be written as follows,

$$G(s) = N(s)/D(s) \tag{16}$$

where $N(s) = s - a_{0(1,1)} - a_{1(1,1)}e^{-\tau_1 s} - a_{2(1,1)}e^{-\tau_2 s}$ and

$$\begin{aligned} D(s) &= s^2 - (a_{0(1,1)} + a_{0(2,2)})s + (a_{0(1,1)} \cdot a_{0(2,2)} - a_{0(1,2)} \cdot a_{0(2,1)}) \\ &-(a_{1(1,1)} + a_{1(2,2)})se^{-\tau_1 s} - (a_{2(1,1)} + a_{2(2,2)})se^{-\tau_2 s} \\ &+(a_{0(2,2)} \cdot a_{1(1,1)} + a_{1(2,2)} \cdot a_{0(1,1)} - a_{0(1,2)} \cdot a_{1(2,1)} - a_{1(1,2)} \cdot a_{0(2,1)})e^{-\tau_1 s} \\ &+(a_{0(2,2)} \cdot a_{2(1,1)} + a_{2(2,2)} \cdot a_{0(1,1)} - a_{0(1,2)} \cdot a_{2(2,1)} - a_{2(1,2)} \cdot a_{0(2,1)})e^{-\tau_2 s} \\ &+(a_{1(1,1)} \cdot a_{1(2,2)} - a_{1(1,2)} \cdot a_{1(2,1)})e^{-2\tau_1 s} \\ &+(a_{2(1,1)} \cdot a_{2(2,2)} - a_{2(1,2)} \cdot a_{2(2,1)})e^{-2\tau_2 s} \\ &+(a_{1(2,2)} \cdot a_{2(1,1)} + a_{2(2,2)} \cdot a_{1(1,1)} - a_{1(1,2)} \cdot a_{2(2,1)} - a_{2(1,2)} \cdot a_{1(2,1)})e^{-(\tau_1+\tau_2)s} \end{aligned}$$

Then the characteristic equation of the closed loop system $C(s)G(s)$ in Fig. 1 can be calculated as,

$$\begin{aligned} \Delta(s) &= (1+k_d)s^3 + (k_p - a_{0(1,1)} - a_{0(2,2)} - a_{0(1,1)} \cdot k_d)s^2 \\ &+(a_{0(2,2)} \cdot a_{0(1,1)} - a_{0(1,2)} \cdot a_{0(2,1)} - a_{0(1,1)} \cdot k_p + k_i)s \\ &-a_{0(1,1)} \cdot k_i - (a_{1(1,1)} + a_{1(2,2)} + a_{1(1,1)} \cdot k_d)s^2 e^{-\tau_1 s} \\ &-(a_{2(1,1)} + a_{2(2,2)} + a_{2(1,1)} \cdot k_d)s^2 e^{-\tau_2 s} \\ &+(a_{0(2,2)} \cdot a_{1(1,1)} + a_{0(1,1)} \cdot a_{1(2,2)} \\ &-a_{0(1,2)} \cdot a_{1(2,1)} - a_{0(2,1)} \cdot a_{1(1,2)} - a_{1(1,1)} \cdot k_p)se^{-\tau_1 s} \\ &+(a_{0(2,2)} \cdot a_{2(1,1)} + a_{0(1,1)} \cdot a_{2(2,2)} \\ &-a_{0(1,2)} \cdot a_{2(2,1)} - a_{0(2,1)} \cdot a_{2(1,2)} - a_{2(1,1)} \cdot k_p)se^{-\tau_2 s} \\ &+(a_{1(1,1)} \cdot a_{1(1,1)} - a_{1(1,2)} \cdot a_{1(2,1)})se^{-2\tau_1 s} \\ &+(a_{2(2,2)} \cdot a_{2(1,1)} - a_{2(2,1)} \cdot a_{2(1,2)})se^{-2\tau_2 s} \\ &+(a_{1(2,2)} \cdot a_{2(1,1)} + a_{1(1,1)} \cdot a_{2(2,2)} \\ &-a_{1(1,2)} \cdot a_{2(2,1)} - a_{1(2,1)} \cdot a_{2(1,2)})se^{-(\tau_1+\tau_2)s} \\ &-a_{1(1,1)} \cdot k_i e^{-\tau_1 s} - a_{2(1,1)} \cdot k_i e^{-\tau_2 s} \end{aligned} \tag{17}$$

The even and odd parts of the numerator and denominator of the transfer function in (16) can be calculated in the form of (9) as follows,

$$N_e = -a_{0(1,1)} - a_{1(1,1)} \cdot \cos(\tau_1 \omega) - a_{2(1,1)} \cdot \cos(\tau_2 \omega) \tag{18}$$

$$N_o = 1 + \frac{a_{1(1,1)}}{\omega} \cdot \sin(\tau_1 \omega) + \frac{a_{2(1,1)}}{\omega} \cdot \sin(\tau_2 \omega) \tag{19}$$

$$\begin{aligned} D_e &= a_{0(1,1)} \cdot a_{0(2,2)} - \omega^2 \\ &+(a_{0(1,1)} \cdot a_{1(2,2)} + a_{0(2,2)} \cdot a_{1(1,1)}) \cdot \cos(\tau_1 \omega) \\ &+(a_{1(2,2)} + a_{1(1,1)}) \cdot \omega \cdot \sin(\tau_1 \omega) \\ &+(a_{1(1,1)} \cdot a_{1(2,2)} + a_{1(1,2)} \cdot a_{1(2,1)}) \cdot \cos(2\tau_1 \omega) \\ &-a_{2(1,2)} \cdot a_{2(2,1)} \cdot \cos(2\tau_2 \omega) \\ &+(a_{1(1,2)} \cdot a_{2(2,1)} - a_{2(1,2)} \cdot a_{1(2,1)}) \cdot \cos((\tau_1 + \tau_2)\omega \end{aligned} \tag{20}$$

$$\begin{aligned} D_o &= a_{0(1,1)} + a_{0(2,2)} + (a_{1(2,2)} + a_{1(1,1)}) \cdot \cos(\tau_1 \omega) \\ &- \frac{(a_{0(1,1)} \cdot a_{1(2,2)} + a_{0(2,2)} \cdot a_{1(1,1)})}{\omega} \sin(\tau_1 \omega) \\ &- \frac{(a_{1(1,1)} \cdot a_{1(2,2)} + a_{1(1,2)} \cdot a_{1(2,1)})}{\omega} \sin(2\tau_1 \omega) \\ &- \frac{(a_{1(1,2)} \cdot a_{2(2,1)} - a_{2(1,2)} \cdot a_{1(2,1)})}{\omega} \sin((\tau_1 + \tau_2)\omega) \\ &+ \frac{a_{2(1,2)} \cdot a_{2(2,1)})}{\omega} \sin(2\tau_2 \omega) \end{aligned} \tag{21}$$

The CRB can be computed in the (k_p, k_i, k_d) parameter space using Definition 2 and substituting (18)-(21) in (12)-(14). Using the Definition 1 and substituting $s = 0$ in the characteristic polynomial of the system in (17), one can obtain the following,

$$k_i(-a_{0(1,1)} - a_{1(1,1)}e^{-\tau_1 s} - a_{2(1,1)}e^{-\tau_2 s}) = 0 \tag{22}$$

Thus, the RRB can be calculated from (22). Similarly, in order to compute the IRB, one can obtain the following equation using Definition 3 and the characteristic equation of the system in (17),

$$(1+k_d)s^3 = 0 \tag{23}$$

The stability regions for the controller can be computed using the RRB, IRB and CRB boundaries. The method gives the explicit formulae corresponding to these boundaries. If one considers that the parameters of the plant include uncertainty, the parameters can be defined within the certain interval as follows,

$$\begin{aligned}
&a_{0(1,1)} \in [\underline{a_{0(1,1)}}, \overline{a_{0(1,1)}}],\ a_{0(1,2)} \in [\underline{a_{0(1,2)}}, \overline{a_{0(1,2)}}]\\
&a_{0(2,1)} \in [\underline{a_{0(2,1)}}, \overline{a_{0(2,1)}}],\ a_{0(2,2)} \in [\underline{a_{0(2,2)}}, \overline{a_{0(2,2)}}]\\
&a_{1(1,1)} \in [\underline{a_{1(1,1)}}, \overline{a_{1(1,1)}}],\ a_{1(1,2)} \in [\underline{a_{1(1,2)}}, \overline{a_{1(1,2)}}]\\
&a_{1(2,1)} \in [\underline{a_{1(2,1)}}, \overline{a_{1(2,1)}}],\ a_{1(2,2)} \in [\underline{a_{1(2,2)}}, \overline{a_{1(2,2)}}]\\
&a_{2(1,1)} \in [\underline{a_{2(1,1)}}, \overline{a_{2(1,1)}}],\ a_{2(1,2)} \in [\underline{a_{2(1,2)}}, \overline{a_{2(1,2)}}]\\
&a_{2(2,1)} \in [\underline{a_{2(2,1)}}, \overline{a_{2(2,1)}}],\ a_{2(2,2)} \in [\underline{a_{2(2,2)}}, \overline{a_{2(2,2)}}]
\end{aligned} \tag{24}$$

If the parameters of the coefficient matrices of transfer function of the plant in (15) includes uncertainty in the form of (24), then all stabilizing controllers obtained from the stabilizing region of the parameter space will satisfy the robust stability of the given interval plant. Thus, a complete set of robust stabilizing controllers for a multiple time delay system with parametric uncertainty can be obtained.

Computation steps of all stabilizing values of the PI and PID controllers:

Step 1: Obtain the transfer function of the multiple time delay system with parametric uncertainty.
Step 2: Calculate the characteristic polynomial of the closed loop system in the form of (8).
Step 3: Determine the RRB and IRB lines using (22) and (23), respectively.
Step 4: Obtain the expressions for k_p, k_d and k_i using (18)-(21) in (12)-(14) and compute CRB.
Step 5: Using IRB, RRB and CRB, compute all stabilizing regions in the (k_p, k_i) parameter space for a fixed value of k_d and compute all stabilizing regions in the (k_p, k_d) parameter space for a fixed value of k_i.

Step 6: Identify the interval of k_d using the (k_p, k_d) parameter space, in which the system still satisfies stability.
Step 7: Draw the IRB, RRB and CRB in the (k_p, k_i, k_d) parameter space while the value of k_d is changing in an identified interval.
Step 8: Investigate the stability regions in the (k_p, k_i, k_d) parameter space and compute all stabilizing *PID* controllers.
Step 9: In order to meet the robust performance specifications of the multiple time delay system with parameter uncertainty structure, draw the CRB planes for all parameter perturbations of the interval plant.
Step 10: Compute IRB-RRB plane using the IRB and RRB lines, obtained from (22) and (23), for the values of k_d in an identified interval.
Step 11: Compute the stability boundaries in the (k_p, k_i, k_d) parameter space using the IRB-RRB plane and CRB planes.
Step 12: Investigate the stability regions in the (k_p, k_i, k_d) parameter space and compute all robust stabilizing PID controllers in the form of (6), which will make the system robustly stable under all parameter perturbations.
Step 13: If only stabilizing values of the PI controller is desired, then repeat the computation steps for $k_d = 0$ and compute stability region in (k_p, k_i) parameter space.

4. CASE STUDY

In this case study, following state space representation of a multiple time delay system is considered,

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0.6 \\ -0.4 & -1 \end{bmatrix} x(t-5) \\
&\quad + \begin{bmatrix} 0 & -0.6 \\ -0.6 & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)
\end{aligned} \tag{25}$$

In this case study, in order to present the advantages of the proposed method, the plant parameters are selected in the unstable region that was previously studied in (Wang et al., 2009; He et al., 2006). The stability area of the plant has been improved in (He et al., 2006). The stability domain of the same system has been further enlarged in (Wang et al., 2009). The purpose of this study is to find all stabilizing values of the *PID* controllers that will make the unstable system robustly stable.

The transfer function of the plant in (25) can be obtained using Section III as follows,

$$G_1(s) = \frac{s + 2 + e^{-5s}}{\begin{aligned} &s^2 + 2.9s + 1.8 + 2se^{-5s} + 2.9e^{-5s} \\ &+1.24e^{-10s} - 0.36e^{-2s} + 0.12e^{-6s} \end{aligned}} \tag{26}$$

Similarly the characteristic polynomial of a closed loop system can be obtained as follows,

$$\begin{aligned}
\Delta(s) &= (1+k_d)s^3 + (2k_d + k_p + 2.9)s^2 \\
&+ (2k_p + k_i + 1.8)s + 2k_i + (k_d + 2)s^2 e^{-5s} \\
&+ (k_p + 2.9)se^{-5s} + 1.24se^{-10s} - 0.36se^{-2s} \\
&+ 0.12se^{-6s} + k_i e^{-5s}
\end{aligned} \tag{27}$$

Even and odd parts of the transfer function of the system can be calculated using (18)-(21). One can derive the expressions for k_p, k_d and k_i using the even and odd parts of the transfer function in (12)-(14). Using (12) and (14), the (k_p, k_i) parameter space can be obtained for a fixed value of k_d and the RRB line can be computed using (22), (see Fig. 2). Similarly the (k_p, k_d) parameter space can be obtained for a fixed value of k_i using (12) and (13), and the IRB line can be computed using (23), (see Fig. 3). The RRB and IRB lines are computed respectively as $k_i = 0$ and $k_d = -1$. One can obtain the stability regions for *PI* and *PD* controllers using the (k_p, k_i) and (k_p, k_d) parameter spaces given in Figs. 2 and 3 respectively.

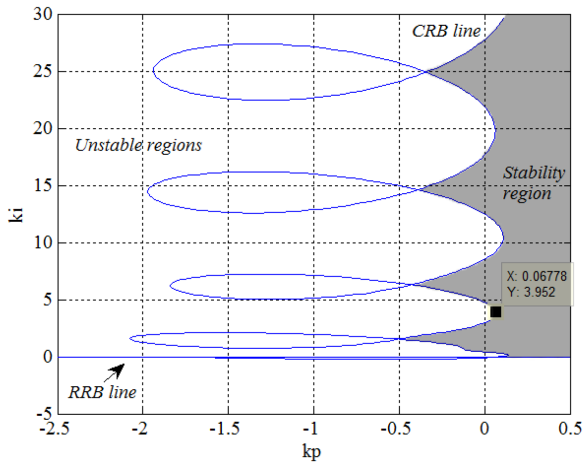

Fig. 2. Stability region in the (k_p, k_i) parameter space for $k_d = 0$.

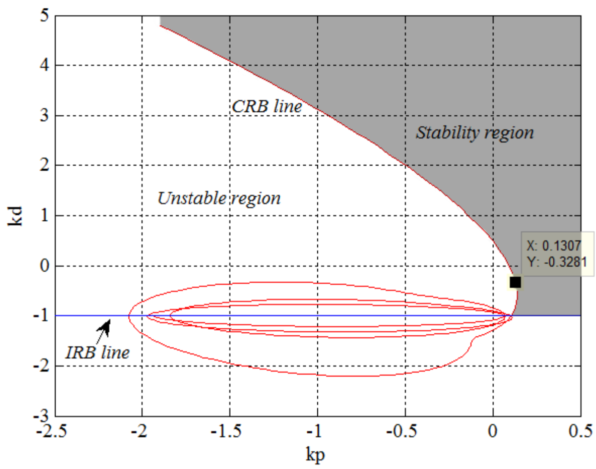

Fig. 3. Stability region in the (k_p, k_d) parameter space for the values $k_i = 0$.

Any point on the CRB lines in Figs. 2 and 3 will show oscillatory responses. For example, let the *PI* and *PD* controllers obtained from the stability region of the Figs. 2 and 3 be $C_{PI}(s)$ and $C_{PD}(s)$, respectively. The step response of the closed loop system $C_{PI}(s)G_1(s)$ shows the oscillatory responses for the values of $k_p = 0.06778$ and $k_i = 3.952$, which are selected on the CRB line in Fig. 2. Similarly, using Fig. 3 the values of $k_p = 0.1307$ and $k_d = -0.3281$ can be selected on the CRB line. The step responses of the $C_{PD}(s)G_1(s)$ system show oscillatory response for these values of *PD* controller. The step responses of the system for these values of the *PI* and *PD* controllers are given in Figs. 4 and 5, respectively.

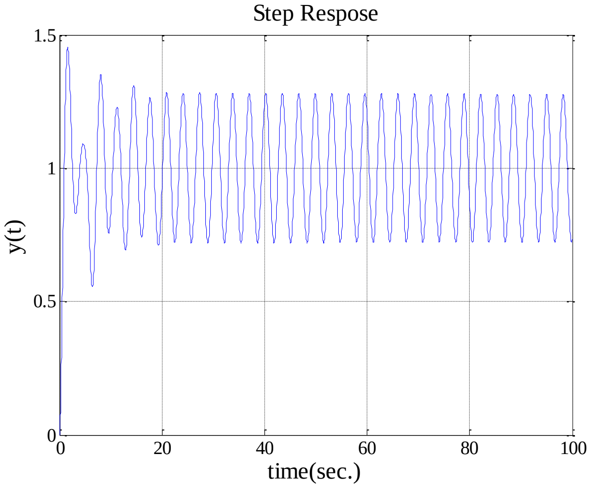

Fig. 4. Step response of the system for the values $k_p = 0.06778$, $k_i = 3.952$ and $k_d = 0$.

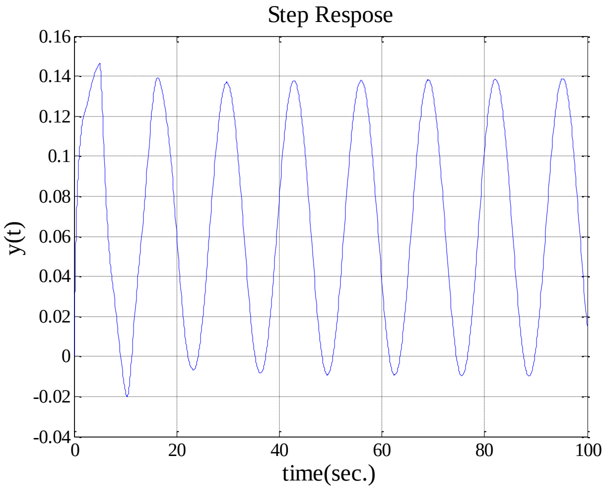

Fig. 5. Step response of the system for the values $k_p = 0.1307$, $k_i = 0$ and $k_d = -0.3281$.

One can conclude from the results of Figs. 2-5 that the right hand side of the CRB lines, namely the shaded region in Figs. 2 and 3, is the stability area for the values of k_p, k_d and k_i while left hand side of the CRB lines remains unstable. Thus, one can find all stabilizing *PI* and *PD* controller parameters for the plant in (25) using Figs. 2 and 3, respectively. On the other hand, all stabilizing values of the *PID* controllers used to control the plant in (25) can be computed in the (k_p, k_i, k_d) parameters space, which is obtained using the expressions of k_p and k_i for the values of k_d within a certain interval. The data range of the k_d can be identified from the results of Fig. 3. One can conclude from Fig. 3 that the CRB line starts at the point $k_d = 4.8$ and decreases as the value of ω increases. The IRB is calculated from (23) as $k_d = -1$, which limits the stability region. Consequently, the value of k_d can be taken in between $-1 < k_d < 4.8$ in which the system still satisfies the stability. Thus, the CRB planes in the (k_p, k_i, k_d) parameter space can be computed using (12) and (14) for the values of k_d in between $-1 < k_d < 4.8$. The IRB and the RRB boundaries constitute the IRB-RRB plane in the (k_p, k_i, k_d) parameter space. The IRB line can be drawn using $[k_p, k_d] = [k_p, -1]$ and the RRB line can be drawn using $[k_p, k_i] = [k_p, 0]$. Thus, the IRB-RRB plane can be computed using $[k_p, k_i] = [k_p, 0]$ lines for the values of k_d between $-1 < k_d < 4.8$ in the (k_p, k_i, k_d) parameter space, where the limit of the plane is $[k_p, k_d] = [k_p, -1]$. Thus, the stability boundaries in the (k_p, k_i, k_d) parameter space can be easily computed using the IRB-RRB plane and CRB plane as shown in Fig. 6. One can easily investigate the stabilizing area for the parameters of the *PID* controllers using (k_p, k_i, k_d) parameter space. The CRB plane divides the (k_p, k_i, k_d) parameter space into stable and unstable regions. The IRB-RRB plane limits the stability region. The $C_1(s)G_1(s)$ system shows oscillatory responses for any point selected on CRB plane. For example, one can compute the *PID* controller in the form of (6), in which the parameters are taken on the CRB plane in Fig. 6 as,

$$C_1(s) = 0.1034 + \frac{12.96}{s} + 0.3s \tag{28}$$

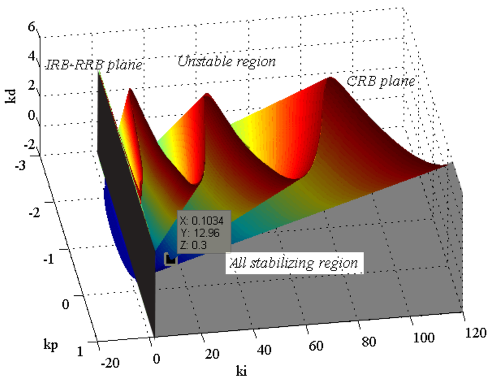

Fig. 6. Stability region for the *PID* controller parameters in the (k_p, k_i, k_d) parameter space.

The step response of the system $C_1(s)G_1(s)$ and the roots of the characteristic polynomial, obtained using the algorithm in (Vyhlídal and Zítek, 2002), are illustrated in Figs. 7 and 8, which shows that the $C_1(s)G_1(s)$ system is oscillatory for these values of the *PID* parameters. Consequently, any points in the stabilizing region of the (k_p, k_i, k_d) parameter space in Fig. 6 satisfy the stability of the system. For example, the step response and the roots of the characteristic polynomial of system for the controller parameters $k_p = 10.1034$, $k_i = 12.96$ and $k_d = 0.3$, which are selected from stability region of the Fig. 6, are illustrated in Figs. 9 and 10, respectively. One can conclude from the results of Figs. 9 and 10 that the controller satisfies the stability of the plant in (25).

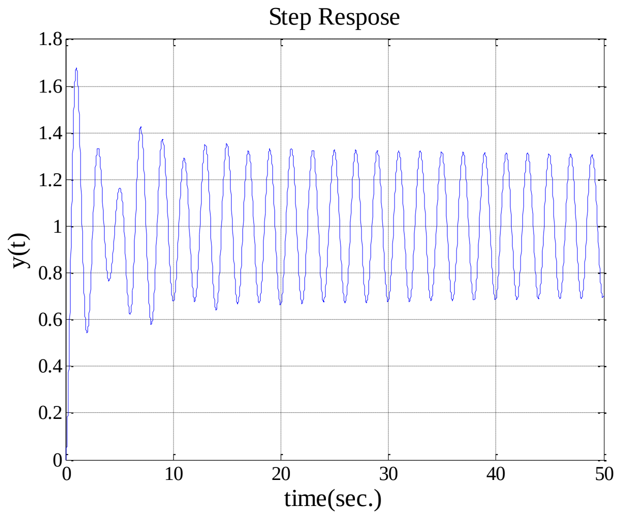

Fig. 7. Step response of the system for $k_p = 0.1034$, $k_i = 12.96$ and $k_d = 0.3$.

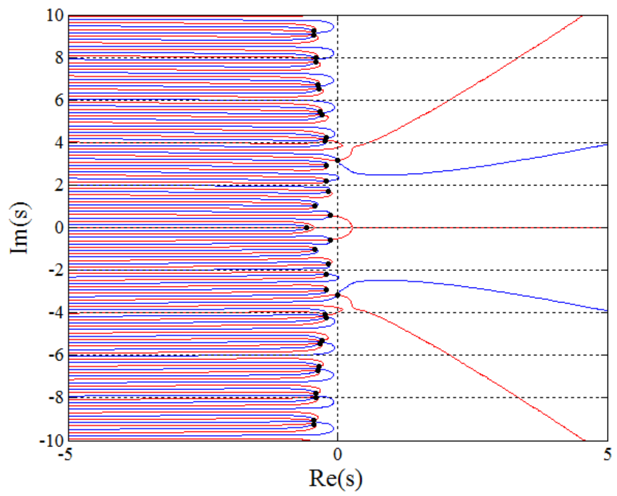

Fig. 8. The roots of the characteristic polynomial for $k_p = 0.1034$, $k_i = 12.96$ and $k_d = 0.3$.

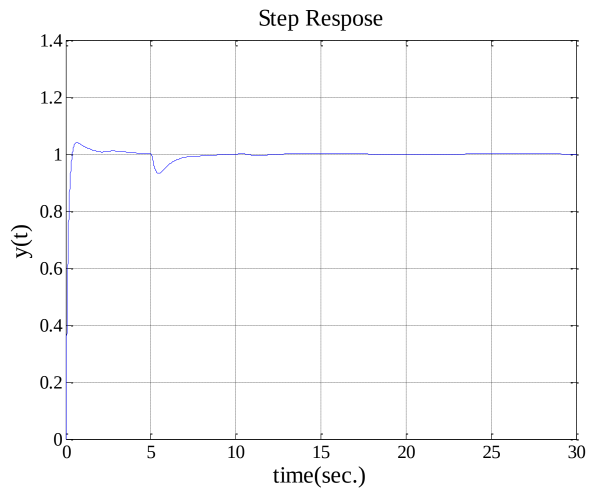

Fig. 9. Step response of the system for $k_p = 10.1034$, $k_i = 12.96$ and $k_d = 0.3$.

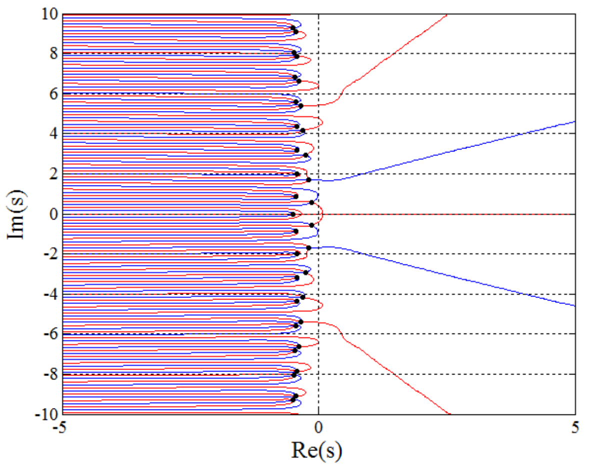

Fig. 10. The roots of the characteristic polynomial for $k_p = 10.1034$, $k_i = 12.96$ and $k_d = 0.3$.

One can extend the above results to multiple time delay systems with parametric uncertainty. The plants with parametric uncertainty are described by a mathematical model containing parameters that are not precisely known, but the values thereof lie within given intervals. This type of uncertainty can arise during the control of real processes. Therefore, modeling of the multiple time delay system, given in (25), with parametric uncertainty is a realistic approach. Consider that the parameter matrices of the system in (25) include uncertainty as follows,

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} [-1.8\ -2.2] & 0 \\ 0 & [-0.7\ -1.1] \end{bmatrix} x(t) \\ &+ \begin{bmatrix} [-0.9\ -1.1] & [0.5\ 0.7] \\ [-0.3\ -0.5] & [-0.9\ -1.1] \end{bmatrix} x(t-5) \\ &+ \begin{bmatrix} 0 & [-0.5\ -0.7] \\ [-0.5\ -0.7] & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)\end{aligned} \tag{29}$$

The parameters of the plant in (29) can be defined in the form of (24). Thus, the transfer function of the system with parametric uncertainty in (29) can be obtained using uncertain parameter matrices of the plant as follows,

$$G_2(s) = N_2(s)/D_2(s) \tag{30}$$

where, $N_2(s) = s - [-2.2, -1.8] - [-1.1, -0.9]e^{-5s}$ and

$$\begin{aligned}D_2(s) &= s^2 - ([-2.2, -1.8] + [-1.1, -0.7])s \\ &+([-2.2, -1.8]\cdot[-1.1, -0.7]) - ([-1.1, -0.9] \\ &+[-1.1, -0.9])se^{-5s} + ([-1.1, -0.7]\cdot[-1.1, -0.9] \\ &+[-1.1, -0.9]\cdot[-2.2, -1.8])e^{-5s} \\ &+([-1.1, -0.9]\cdot[-1.1, -0.9] - [0.5, 0.7]\cdot[-0.5, -0.3])e^{-10s} \\ &-([-0.7, -0.5]\cdot[-0.7, -0.5])e^{-2s} \\ &-([0.5, 0.7]\cdot[-0.7, -0.5] + [-0.7, -0.5]\cdot[-0.5, -0.3])e^{-6s}\end{aligned}$$

Similarly, the characteristic polynomial of the system can be obtained using (17) for the uncertain parameters of the system. The CRB planes of the system in the parameter space of the controller can be computed for all parameter perturbations of the plant in (30). Consequently, $3^8 = 6561$, CRB planes can be drawn using three values for each of the eight uncertain parameters of the plant in (30). The IRB-RRB plane, which limits the stability boundary, can also be computed as previously explained. Then, all stabilizing areas can be investigated in the parameter space. One can easily select the parameters of all stabilizing controllers, which will make the system robustly stable under all parameter perturbations of the plant, using the stability regions in the (k_p, k_i, k_d) parameter space.

One can easily compute the stability boundaries, namely RRB and CRB lines, in the parameter space of *PI* controllers under all parameter perturbations of the plant, as shown in Fig. 11. Similarly, the stability boundaries, namely IRB-RRB plane and CRB planes, can be computed in the parameter space of the *PID* controllers as given in Fig. 12.

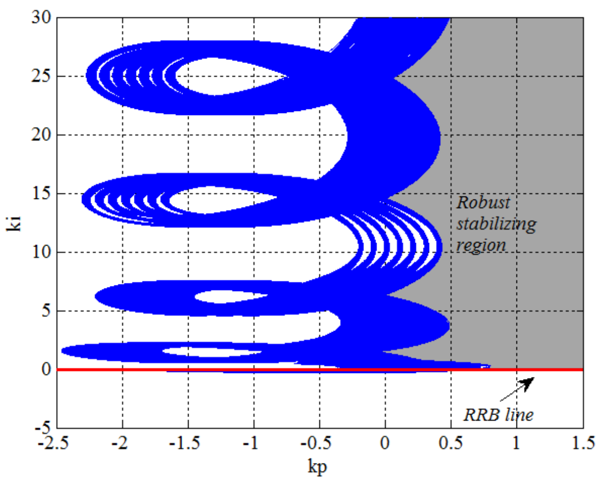

Fig. 11. Robust stability region for a *PI* controller in the (k_p, k_i) parameter space.

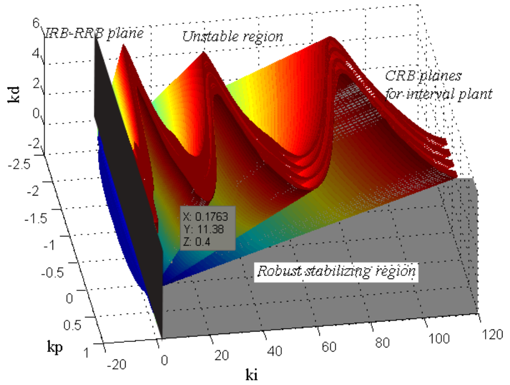

Fig. 12. Robust stability region for a PID controller in the (k_p, k_i, k_d) parameter space.

Consequently, all stabilizing robust *PI* and *PID* controllers can be obtained under all parameter perturbations of the plant in (30) using stabilizing regions in Figs. 11 and 12 respectively. In other words, any values of the *PI* and *PID* controllers selected from any point in the robust stabilizing region, namely the shaded region in Figs. 11 and 12 respectively, will stabilize the multiple time delay system with parametric uncertainty under all parameter perturbations of the plant. On the other hand, any controller selected on the CRB lines in Fig. 11 or CRB planes in Fig. 12 will show oscillatory response to related transfer functions. Let $C_2(s)$ be controller, which parameters are selected form stability region of the parameter space of the *PID* controller in Fig. 12 as follows,

$$C_2(s) = 10.17 + \frac{11.38}{s} + 0.4s \tag{31}$$

Step responses of the $C_2(s)G_2(s)$ system can be computed under all parameter perturbations of the plant using the controller with the transfer functions of the plant in (30). Thus, $3^6 = 729$ step responses can be obtained for the system $C_2(s)G_2(s)$ using three values for each of the six uncertain parameter of the plant. The parameters $a_{1(1,1)}$ and $a_{1(2,2)}$ have nominal values. Fig. 13 shows that the *PID* controller in (31) satisfies the robust performances of the system under all parameter perturbations of the plant $G_2(s)$.

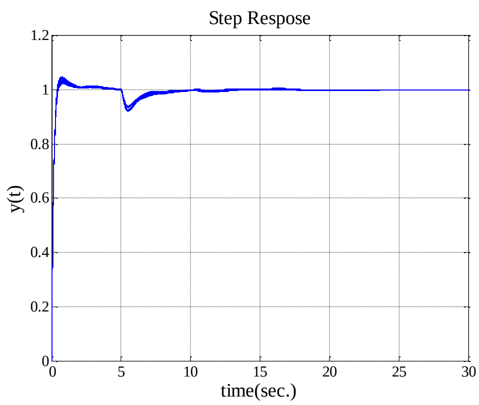

Fig. 13. Step responses of the system $C_2(s)G_2(s)$ for 729 different values of the plant.

5. CONCLUSION

This study is dedicated to investigation of the stability region of the controller parameters for multiple time delay systems with parametric uncertainty, due to its importance in control applications. Stabilizing controller approach has been extended to compute all stabilizing *PI* and *PID* controllers. The key points of the proposed method are given as,

- All stabilizing *PI* and *PID* controllers are computed using the parameter space,
- The method provides a simple and effective way of computing stability region of the parameter space of the controller that guaranties the robust stability of the multiple time delay system with parameter uncertainty structure.

The results of the paper may be discussed for different applications. For example:

- Stabilizing controller design for unstable multiple time delay systems and large time delay systems may be a challenging subjects.
- Recent researches on stabilizing controller design for discrete systems shows that the discrete time controllers may also deserve new discussions. Recently, a graphical technique for finding all discrete-time *PID* controllers that satisfy the robust stability constraint was proposed in (Emami and Hartnett, 2014). The stability boundary and the number of unstable poles in the integral derivative plane for continuous-time or discrete-time *PID* controllers were investigated in (Emami et al., 2011). A parameter space approach for designing digital *PID* controllers is studied in (Kiani and Bozorg, 2006). Thus, the study on discrete-time *PID* controllers that satisfy the robust stability constraint for multiple time delay systems may also be a promising study.

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