Turk J Math
33 (2009) , 265 – 272.

doi:10.3906/mat-0711-31

On the geometry of null curves in the minkowski 4-space

R. Aslaner, A. İhsan Boran

Abstract

In this paper, we study the basic results on the general study of null curves in the Minkowski 4-space R_1^4. A transversal vector bundle of a null curve in R_1^4 is constructed using a frenet Frame consisting of two real null and two space-like vectors. The null curves are characterized by using the Frenet frame.

Key Words: Null curves, Minkowski space, Transversal vector bundle.

1. Introduction

Definition 1.1 *The Minkowski 4-space is the space R^4 with the Lorentzian inner product*

$$g(x, y) = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3 \qquad \text{for all } \; x, y \in R^4$$

and will be denoted in the future by R_1^4. With respect to the standard basis of R_1^4, the matrix of g is $\eta = diag(-1, 1, 1, 1)$.

Definition 1.2 *A non-zero vector x of R_1^4 is called space-like if $g(x, x) > 0$, time-like if $g(x, x) < 0$, null if $g(x, x) = 0$ and causal if $g(x, x) \leq 0$. Any two vectors $x, y \in R_1^4$ are called orthogonal if $g(x, y) = 0$. The zero vector is taken to be space-like.*

Lemma 1.1 *[4] There are no casual vectors in R_1^4 orthogonal to a time-like vector and two null vectors are orthogonal if and only if they are linearly dependent.*

Let C be a smooth curve in R_1^4 with the immersion $i : C \longrightarrow R_1^4$. Suppose U is a coordinate neighborhood on C and t is the corresponding local parameter. Then C is given by the map

$$\begin{array}{ccccccc} C & : & I & \longrightarrow & R_1^4 & & \\ & & t & \longrightarrow & C(t) & = & (x^0(t), x^1(t), x^2(t), x^3(t)) \end{array}$$

where I is an open interval of R. The tangent vector field on U of C is

$$T = \frac{dC}{dt} = \left(\frac{dx^0}{dt}, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right). \tag{1.1}$$

The smooth curve C is said to be a *null (light-like or isotropic) curve* if the tangent vector to C at any point is a null vector. It follows that C is a null curve if and only if locally at each point of U we have

$$g(T,T)=0 \tag{1.2}$$

2. Null curves in minkowski 4-space R_1^4

Let C be a null curve in R_1^4, that is, $T=\dfrac{dC}{dt}$ and $g(T,T)=0$. Denote by TC the tangent bundle of C and define

$$TC^{\perp}=\underset{p\in C}{\cup}T_pC^{\perp}\ ;\qquad T_pC^{\perp}=\left\{v_p\in R_1^4: g(v_p,\xi_p)=0\right\}$$

[2], where ξ_p is a null vector tangent to C at the point P. Clearly $TC^{\perp}$ is a vector bundle over C of rank 3. Since $g(\xi_p,\xi_p)=0$, the tangent bundle TC of C is a vector subbundle of $TC^{\perp}$, of rank 1. Consider a complementary vector bundle $S\left(TC^{\perp}\right)$ to TC in $TC^{\perp}$. Thus we have the orthogonal decomposition

$$TC^{\perp}=TC\perp S\left(TC^{\perp}\right),$$

where the fibers of $S\left(TC^{\perp}\right)$ at $P\in C$ are nothing but some screen subspaces of $T_pC^{\perp}$. The vector bundle $S\left(TC^{\perp}\right)$ is called the *screen vector bundle* of C. It follows that $S\left(TC^{\perp}\right)$ is a non-degenerate vector bundle. Therefore we have

$$TR_1^4|_C=S\left(TC^{\perp}\right)\perp S\left(TC^{\perp}\right)^{\perp}, \tag{2.1}$$

where $S\left(TC^{\perp}\right)^{\perp}$ is a complementary orthogonal vector bundle to $S\left(TC^{\perp}\right)$ in $TR_1^4|_C$. We denote the set of sections of TR_1^4 by $\Gamma(TR_1^4)$, that is, the set of the vector fields on TR_1^4. It is important to observe that $\Gamma(TR_1^4)$ is a module over the ring of smooth functions $C^{\infty}(R_1^4)$ on R_1^4.

We recall that a sum of two subspaces is a direct sum if and only if the intersection of the subspaces is $\{0\}$. Then we have the following corollary.

Corollary 2.1 *Let C be a null curve in the Minkowski 4-space R_1^4, and $S\left(TC^{\perp}\right)$ be a screen vector bundle of C. Then the following assertions are equivalent:*

1. $S\left(TC^{\perp}\right)$ is a non-degenerate subbundle,
2. $S\left(TC^{\perp}\right)^{\perp}$ is a non-degenerate subbundle,
3. $S\left(TC^{\perp}\right)$ and $S\left(TC^{\perp}\right)^{\perp}$ are complementary orthogonal vector bundles of $\Gamma(TR_1^4)$
4. $\Gamma(TR_1^4)$ is the orthogonal direct sum of $S\left(TC^{\perp}\right)$ and $S\left(TC^{\perp}\right)^{\perp}$.

A. Bejancu stated and proved the following theorem in [2, Theorem 1.1], which shows the existence and uniqueness of a vector bundle and plays an important role in studying the geometry of null curves.

Theorem 2.1 *Let C be a null curve in the Minkowski 4-space R_1^4, and $S\left(TC^{\perp}\right)$ be a screen vector bundle of C. Then there exists a unique vector bundle E over C of rank 1, such that on each coordinate neighborhood $U \subset C$ there is a unique vector bundle $N \in \Gamma(E\,|_U\,)$ satisfying*

$$g(T,N)=1 \text{ and } g(N,N)=g(N,X)=0,\ \forall X \in \Gamma(S\left(TC^{\perp}\right)|_U\,). \tag{2.2}$$

The vector bundle E is denoted by $ntr(C)$ and called the *null transversal bundle* of C with respect to $S\left(TC^{\perp}\right)$. Next, consider the vector bundle

$$tr(C) = ntr(C) \perp S\left(TC^{\perp}\right),$$

which is complementary but not orthogonal to call TC in TR_1^4. $tr(C)$ the *transversal vector bundle* of C with respect to $S\left(TC^{\perp}\right)$. The vector field N given in Theorem 2.1 is called the *null transversal vector field* of C with respect to T. More precisely, we have

$$TR_1^4\,|_C\; = TC \oplus tr(C) = (TC \oplus ntr(C)) \perp S\left(TC^{\perp}\right) \tag{2.3}$$

As $\{T,N\}$ is a basis of $\Gamma\left((TC \oplus ntr(C))\,|_U\,\right)$, the local vector fields

$$W^{+} = \frac{1}{\sqrt{2}}\{T+N\} \quad \text{and} \quad W^{-} = \frac{1}{\sqrt{2}}\{T-N\}$$

form an orthonormal basis with signature $\{1,-1\}$. Then, it follows that the fibers of $TC \oplus ntr(C)$ are hyperbolic planes with respect to g.

Thus, we can obtain the following proposition.

Proposition 2.1 *Let C be a null curve in R_1^4. Then any screen vector bundle $S\left(TC^{\perp}\right)$ of C is Riemannian.*

Suppose C is a null curve in R_1^4 and D is the Levi-Civita connection on R_1^4. In this case, $\{T,N,W_1,W_2\}$ is a frame along C, where T and N are null vectors and W_1, W_2 are space-like vectors. Then, we obtain the following equations:

$$\begin{array}{lcl} D_T T &=& hT + k_1 W_1 \\ D_T N &=& -hN + k_2 W_1 + k_3 W_2 \\ D_T W_1 &=& -k_2 T - k_1 N + k_4 W_2 \\ D_T W_2 &=& -k_3 T - k_4 W_1, \end{array} \tag{2.4}$$

where h and $\{k_1,k_2,k_3,k_4\}$ are smooth functions on $U \subset C$ and $\{W_1,W_2\}$ is a certain orthonormal basis of $\Gamma\left(S\left(TC^{\perp}\right)|_U\,\right)$. We call $F = \{T,N,W_1,W_2\}$ a Frenet frame on R_1^4 along C with respect to the screen vector bundle $S\left(TC^{\perp}\right)$ and the functions $\{k_1,k_2,k_3,k_4\}$ curvature functions of C with respect to F. Finally, equations (2.4) are called the *Frenet equations* with respect to F [2]

Thus, we may give the following remarks.

Remark 2.1 *For a null curve C in the Minkowski 4-space R_1^4 there always exist a screen vector bundle $S\left(TC^{\perp}\right)$ and a Frenet frame F induced by $S\left(TC^{\perp}\right)$ on any coordinate neighborhood $U \subset C$. In fact, there* *exist a Riemannian metric g on the vector bundle $TC^{\perp}$ over C. Then consider $S\left(TC^{\perp}\right)$ as the complementary orthogonal vector bundle to TC in $TC^{\perp}$ with respect to g.*

If $h = 0$ in (2.4), then the parameter t is said to be a *distinguished parameter*. When we choose t as a distinguished parameter, the first two equations in (2.4) become

$$\begin{aligned} D_T T &= k_1 W_1 \\ D_T N &= k_2 W_1 + k_3 W_2 \end{aligned} \tag{2.5}$$

The other equations remain unchanged. Thus we make the following remarks.

Remark 2.2 *If C is a null curve in R_1^4 given by the distinguished parameter t, then $D_T T$ is a space-like vector field, so we may choose W_1 as a unit space-like vector field collinear to $D_T T$.*

Remark 2.3 *If $k_1 = 0$ in (2.5), then C is a null geodesic in R_1^4.*

Remark 2.4 *Let C be a null curve given by the distinguished parameter in R_1^4. Then C is a null geodesic of R_1^4 if and only if the first curvature k_1 vanishes identically on C.*

Definition 2.1 *Assume that $C \subset R_1^4$ is a null curve with curvature functions k_1, k_2, k_3. Then the harmonic functions of C in R_1^4 are defined as*

$$H_i = \begin{cases} \dfrac{k_1}{k_2}\ ; & if \quad i = 1 \\[2ex] \dfrac{H_1'}{k_3}\ ; & if \quad i = 2. \end{cases}$$

Example 2.1 *Consider the curve C in R_1^4 given by the equation*

$$C(t) = \frac{1}{\sqrt{2}}(\sinh t, \cosh t, \sin t, \cos t), \quad t \in R$$

Then the tangent vector bundle of C is

$$T = \frac{dC}{dt} = \frac{1}{\sqrt{2}}(\cosh t, \sinh t, \cos t, -\sin t)$$

Since $g(T, T) = 0$, C is a null curve. Moreover,

$$D_T T = \frac{1}{\sqrt{2}}(\sinh t, \cosh t, -\sin t, -\cos t) \quad and \quad g(D_T T, D_T T) = 1 > 0,$$

$D_T T$ is a space-like vector field, so we can take $D_T T = W_1$ which implies that $h = 0$ and $k_1 = 1$ in the first equation of (2.4). Thus $h = 0$ implies that t is the distinguished parameter for C and by Remark 2.3, C is a non-geodesic in R_1^4. By taking the derivative of W_1 with respect to T, we have

$$D_T W_1 = \frac{1}{\sqrt{2}}(\cosh t, \sinh t, -\cos t, \sin t,) \tag{2.6}$$

Choosing $W_2 = \frac{1}{\sqrt{2}}(\sinh t, \cosh t, \sin t, \cos t)$, *and taking the derivative with respect to* T, *we have*

$$D_T W_2 = \frac{1}{\sqrt{2}}(\cosh t, \sinh t, \cos t, -\sin t,) = T.$$

This implies that $k_3 = -1,\ k_4 = 0$ *in equation (2.4) and we obtain*

$$N = \frac{1}{\sqrt{2}}(-\cosh t, -\sinh t, \cos t, -\sin t,).$$

By taking the derivative of N *with respect to* T, *we have*

$$D_T N = \frac{1}{\sqrt{2}}(-\sinh t, -\cosh t, \cos t, -\sin t,) = -W_2.$$

This implies that $k_2 = 0$ *in equation (2.4), so the harmonic curvatures* H_1 *and* H_2 *of* C *are indefinite.*

3. The characterizations of null helices in minkowski 4-space R_1^4

In the Euclidean space R^3, a helix satisfies that its tangent makes a constant angle with a fixed direction called the axis. In the general case, we must replace the fixed direction by a parallel vector field. The authors proved that a curve is a helix if and only if there exists a parallel vector field lying in the osculating space of the curve and making constant angles with the tangent and the principal normal [5].

When the ambient space is a Minkowski space, then some results have been obtained. For example, in [3] a non-null curve α immersed in R_1^3 is a helix if and only if its tangent indicatrix is contained in some plane.

In the geometry of null curves difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. A method of proceeding is to introduce a new parameter called the pseudo-arc which normalizes the derivative of the tangent vector.

Suppose C is a null curve in R_1^4 given by the distinguished parameter. Moreover, if the last curvature k_4 vanishes, then the frame $\{T, N, W_1, W_2\}$ is called a *distinguished Frenet frame* [6].

Definition 3.1 *Let* $C : I \subset R \longrightarrow R_1^4$ *be a null curve and* X *be a non-zero constant vector field in* R_1^4. *If* $g(T, X) \neq 0$ *is a constant for all* $t \in I$, *then the curve* C *is said to be* null helix *and* $sp\{X\}$ *is said to be the* inclination axes *of* C.

Let C be a null helix give by the distinguished parameter and $\{T, N, W_1, W_2\}$ be a distinguished Frenet frame in R_1^4. If C is a non geodesic curve, then there exists a unit constant vector field X such that $g(T, X) = constant$. Thus by taking the derivative we obtain $g(D_T T, X) = 0$. Moreover, by using the first equation from (2.5) we obtain

$$g(D_T T, X) = k_1 g(W_1, X). \tag{3.1}$$

Since $g(D_T T, X) = 0$ and from (2.5) $k_1 \neq 0$, we may write

$$g(W_1, X) = 0. \tag{3.2}$$

By taking the derivative of (3.2) with respect to T and using the third equation in (2.4), we have

$$g(D_T W_1, X) = 0 \Longrightarrow -k_2 g(T, X) - k_1 g(N, X) = 0$$

and

$$g(N, X) = -\frac{k_2}{k_1} g(T, X) = -H_1^{-1} g(T, X), \tag{3.3}$$

where H_1 is the first harmonic curvature of the curve C. By taking the derivative of (3.3) with respect to T and using the second equation in (2.5), we have

$$g(D_T N, X) = \frac{H_1{}'}{H_1^2} g(T, X) = k_3 g(W_2, X).$$

This implies that

$$g(W_2, X) = \frac{H_1{}'}{H_1^2 k_3} g(T, X) = \frac{H_2}{H_1^2} g(T, X), \tag{3.4}$$

where H_2 is the second harmonic curvature of the curve C. By taking the derivative of (3.4) with respect to T and using the last equation in (2.4) we have

$$g(D_T W_2, X) = \left(\frac{H_2{}'}{H_1^2} - \frac{2H_1{}' H_2}{H_1^3} \right) g(T, X) = -k_3 g(T, X)$$

$$\Longrightarrow H_2{}' = \left(\frac{2H_2^2}{H_1} - H_1^2 \right) k_3.$$

Thus we can state the following theorem.

Theorem 3.1 *If C is a null helix given by a distinguished Frenet frame $\{T, N, W_1, W_2\}$ and curvature functions k_1, k_2, k_3, then there exists a unit constant vector field X in R_1^4 such that, $sp\{X\}$ being a slope axis,*

$$g(N, X) = -H_1^{-1} g(T, X), \quad g(W_1, X) = 0, \quad g(W_2, X) = \frac{H_2}{H_1^2} g(T, X),$$

where H_1 and H_2 are the first and second harmonic curvatures of C, respectively, and

$$H_2{}' = \left(\frac{2H_2^2}{H_1} - H_1^2 \right) k_3.$$

Example 3.1 *Let $C : I \subset R \longrightarrow R_1^4$ be the null curve defined by*

$$C(t) = (t, 0, \cos t, \sin t),$$

and $X = (1, 0, 0, 0)$ a unit constant vector field in R_1^4. The tangent vector bundle of C is

$$T = \frac{dC}{dt} = (1, 0, -\sin t, \cos t).$$

C is a null helix since $g(T,T)=0$ and $g(T,X)=-1=constant$.
Also the frame $\{T,N,W_1,W_2\}$ is a distinguished Frenet frame along C where

$N=\frac{1}{2}(-1,0,-\sin t,\cos t)$,

$W_1=(0,0,-\cos t,-\sin t)$,

$W_2=(0,1,0,0)$.

Thus we can find the results

$k_1=1$, $k_2=\frac{1}{2}$, $k_3=0$ *and* $H_1=2$, $H_2=0$.

Example 3.2 *Let C be the curve in R_1^4 defined by*

$$C(t)=(\frac{1}{3}t^3+2t,t^2,\frac{1}{3}t^3,2t),\quad t\in R$$

Then $\dfrac{dC}{dt}=T=(t^2+2,2t,t^2,2)$ and $g(T,T)=0$, so C is a null curve in R_1^4. If we take $X=(0,0,0,\frac{1}{2})$, then $g(T,X)=1\neq 0$ is a constant. Therefore the curve C is a null helix. Since $D_TT=2(t,1,t,0)\neq 0$, C is a non geodesic curve. Thus

$$N=-\frac{1}{8}(t^2+2,2t,t^2,-2).$$

Since $h=g(D_TT,N)=0$, the parameter t is a distinguished parameter for C. Hence from (2.4) we have $k_1=2$ and

$$W_1=(t,1,t,0).$$

Since $D_TN=-\dfrac{1}{4}(t,1,t,0)=-\dfrac{1}{4}W_1$ we have $k_2=-\dfrac{1}{4}$ and $k_3=0$. Choose

$$W_2=\frac{1}{2}(t^2,2t,t^2-2,0)$$

then $D_TW_2=W_1$ and $D_TW_1=\dfrac{1}{4}T-2N-W_2$.

The harmonic functions of C are $H_1=\dfrac{k_1}{k_2}=-8$ and $H_2=0$.

Theorem 3.2 *A null curve C is a helix in the Minkowski 4-space R_1^4 if and only if there exists a parallel vector field lying in the space $Sp\{T,N,W_2\}$ of the curve; orthogonal to W_1 and making constant angles with T which is the tangent of C.*

Proof: The necessary part follows from Theorem 3.1. So we prove the sufficient part. Let X be a non-zero constant vector field in the space $sp\{T,N,W_2\}$, hence $g(W_1,X)=0$. Clearly this implies that

$$k_1g(W_1,X)=0$$

Since $k_1g(W_1,X)=g(D_TT,X)$ we get $g(T,X)$ is a constant. Therefore C is a null helix.

References

[1] Aslaner R.: Hyperruled Surfaces in Minkowski 4-space, IRAN J SCI TECHNOL. A, Vol. 29(A3), 341-347 (2005).

[2] Duggal K.L. and Bejancu A.: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Academic Publisher. Dordrecht / Boston / London (1996).

[3] Ferrandez A., Gimenez A. and Lucas P.: Null Helices in Lorentzian Space Forms, INT J MOD PHYS A, 16(30) 4845-4863 (2001).

[4] Garcia-Rio E. and Kpeli D.N.: Semi-Riemannian Maps and Their Applications, Kluwer Academic Publisher. Dordrecht / Boston / London (1999).

[5] Graves L.: Codimension one Isometric Immersions between Lorentz Spaces, Trans. Amer. Math. Soc. 367-392 (1979).

[6] Yalınız A.F. and Hacısalihoğlu H.H.: Null Generalized Helices in L_3 and L_4, 3 and 4-dimensional Lorentzian Space, Math. And Comp. Appl. 10(1), 105-111 (2005).

R. ASLANER, A. İHSAN BORAN
Department of Mathematics,
Faculty of Education,
İnönü University,
44280 Malatya-TURKEY
e-mail: raslaner@inonu.edu.tr

Received 27.11.2007