



## Generalized Difference Sequence Spaces Defined by Orlicz Functions

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### Abstract

The idea of difference sequence space was introduced by Kızmaz [1], and this concept was generalized by Et-Çolak [2]. In this paper we define some generalized difference sequence spaces by using an Orlicz function and examine some properties of these spaces.

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**Key words :** Difference sequence, Orlicz function.

## Orlicz Fonksiyonu Yardımıyla Tanımlanan Genelleştirilmiş Fark Dizi Uzayları

### Özet

Fark dizi uzayları Kızmaz [1] tarafından tanımlandı, bu fikir Et-Çolak [2] tarafından genelleştirildi. Bu makalede bir Orlicz fonksiyonu kullanarak genelleştirilmiş fark dizi uzaylarını tanımladık ve bu dizi uzaylarının bazı özelliklerini inceledik.

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**Anahtar Kelimeler :** Fark diziler, Orlicz fonksiyonu.

### 1. Introduction

Let  $w$  denotes the space of all complex sequences and  $l_\infty$ ,  $c$ , and  $c_0$  be the linear spaces of bounded, convergent, and null sequences  $x = (x_k)$  with complex terms, respectively, normed by  $\|x\|_\infty = \sup_k |x_k|$ , where  $k \in N = \{1, 2, \dots\}$ , the set of positive integers.

The difference sequence space  $X(\Delta)$  was introduced by Kızmaz [1] as follows :

$$X(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in X\},$$

for  $X = l_\infty, c$  and  $c_0$ ; where  $\Delta x_k = x_k - x_{k+1}$ , for all  $k \in N$ .

The notion of difference sequence spaces was further generalized by Et and Colak [2] as follows:

$$X(\Delta^n) = \{x = (x_k) \in w : (\Delta^n x_k) \in X\},$$

for  $X = l_\infty, c$  and  $c_0$ ; where  $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$  and  $\Delta^0 x_k = x_k$  for all  $k \in N$ . Taking  $X = l_\infty(p), c(p)$  and  $c_0(p)$ , these sequence spaces has been generalized by Et and Başarı [3].

The generalized difference has the following binomial representation :

$$\Delta^n x_k = \sum_{\nu=0}^n (-1)^\nu \binom{n}{\nu} x_{k+\nu} \quad , \text{ for all } k \in \mathbb{N}.$$

The following inequality will be used throughout the paper. Let  $p = (p_k)$  be a positive sequence of real numbers with  $0 < p_k \leq \sup_k p_k = G$ ,  $C = \max(1, 2^{G-1})$ .

Then for  $a_k, b_k \in \mathbb{C}$  for all  $k \in \mathbb{N}$ , we have

$$\left| a_k + b_k \right|^{p_k} \leq C \left\{ \left| a_k \right|^{p_k} + \left| b_k \right|^{p_k} \right\}, \text{ for all } k \in \mathbb{N}, \quad [4; \text{p:346}] \quad (1)$$

Throughout the paper, we shall write  $\sum_k$  for  $\sum_{k=1}^{\infty}$ ,  $\lim_n$  for  $\lim_{n \rightarrow \infty}$  and by  $X$  we shall denote one of the sequence spaces  $\ell_\infty, c$  and  $c_0$

An Orlicz function is a function  $M: [0, \infty) \rightarrow [0, \infty)$ , which is continuous, non decreasing and convex with  $M(0)=0$ ,  $M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

An Orlicz function  $M$  is said to satisfy  $\Delta_2$ -condition for all values of  $u$ , if there exists a constant  $K > 0$ , such that  $M(2u) \leq KM(u)$  ( $u \geq 0$ ). The  $\Delta_2$ -condition is equivalent to  $M(Lu) \leq KLM(u)$  for all values of  $u$  and for  $L \geq 1$ .

Lindenstrauss and Tzafriri [5] used the notion of Orlicz function to construct the sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

become a Banach space, which is called an Orlicz sequence space. The space  $\ell_M$  is closely related to the space  $\ell^p$ , which is an Orlicz sequence space with  $M(x) = x^p$  for

$$1 \leq p < \infty.$$

A sequence space  $E$  is said to be *solid* (or *normal*) if  $(\alpha_k x_k) \in E$ , whenever  $(x_k) \in E$  and for all sequence  $(\alpha_k)$  of scalars such that  $|\alpha_k| \leq 1$  for all  $k \in \mathbb{N}$ .

A sequence space  $E$  is said to be *monotone* if  $E$  contains preimages of all its step spaces

A sequence space  $E$  is said to be *sequence algebra* if  $x.y \in E$  whenever  $x, y \in E$  [6; p:46,47,48].

It is well known that if  $M$  is a convex function and  $M(0) = 0$ , then  $M(\lambda x) \leq \lambda M(x)$  for all  $\lambda$  with  $0 < \lambda < 1$ .

**Lemma 1** A sequence space E is solid implies E is monotone.

Mursaleen et al [7] defined the sequence spaces  $c_o(\Delta, M, p)$ ,  $c(\Delta, M, p)$  and  $l_\infty(\Delta, M, p)$  and investigated some properties of these sequence spaces

Let  $p = (p_k)$  be any sequence of positive real numbers and n be a positive integer. We define the following sequence sets:

$$c_o(\Delta^n, M, p) = \left\{ x = (x_k) : \lim_{k \rightarrow \infty} \left[ M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right]^{p_k} = 0, \text{ for some } \rho > 0 \right\},$$

$$c(\Delta^n, M, p) = \left\{ x = (x_k) : \lim_{k \rightarrow \infty} \left[ M \left( \frac{|\Delta^n x_k - l|}{\rho} \right) \right]^{p_k} = 0, \text{ for some } \rho > 0, l \in C \right\},$$

$$l_\infty(\Delta^n, M, p) = \left\{ x = (x_k) : \sup_{k \geq 0} \left[ M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

In the case  $p_k = \text{constant}$  for all  $k$ , we denote the above sequence spaces as  $c_o(\Delta^n, M)$ ,  $c(\Delta^n, M)$  and  $l_\infty(\Delta^n, M)$ , respectively. If we take  $n = 1$ , then we get the sequence spaces  $c_o(\Delta, M, p)$ ,  $c(\Delta, M, p)$  and  $l_\infty(\Delta, M, p)$  from the above sequence spaces.

## 2. Main Results

**Theorem 2.1.** Let  $(p_k)$  be bounded, n be a positive integer and M be an Orlicz function. Then  $X(\Delta^{n-1}, M, p) \subset X(\Delta^n, M, p)$  and the inclusion is strict.

**Proof.** Since M is non decreasing and convex, the result follows by (1). To show that the inclusion is strict, let  $M(x) = x$ , and  $p_k = 1$  for all  $k \in \mathbb{N}$ , then the sequence  $x = (k^m)$  belongs to  $l_\infty(\Delta^n, M, p)$ , but does not belong to  $l_\infty(\Delta^{n-1}, M, p)$ .

It is easy to show that these sequence spaces are paranormed spaces with

$$G(x) = \inf \left\{ \rho^{p_m/H} : \left\{ \sup_{k \geq 0} \left( M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right)^{p_k} \right\}^{1/H} \leq 1 \right\} \quad (2)$$

where  $H = \max(1, \sup_{k \geq 0} p_k)$ .

**Theorem 2.2.**  $l_\infty(\Delta^n, M, p)$  is a complete paranormed spaces with  $G$  defined in (2).

**Proof:** Using the same technique of Theorem 2.1. given by Murseleen et al [7], it is easy to prove the theorem.

**Theorem 2.3.** Let  $0 < p_k \leq q_k < \infty$  for each  $k$ . Then  $c_0(\Delta^n, M, p) \subset c_0(\Delta^n, M, q)$ .

**Proof.** Proof is trivial.

**Theorem 2.4.(i)** Let  $0 < \inf p_k \leq p_k \leq 1$ . Then  $c_0(\Delta^n, M, p) \subset c_0(\Delta^n, M)$ .

**(ii)** Let  $1 \leq p_k \leq \sup p_k < \infty$ . Then  $c_0(\Delta^n, M) \subset c_0(\Delta^n, M, p)$ .

**Proof. (i)** Let  $x \in c_0(\Delta^n, M, p)$ , that is

$$\lim_{k \rightarrow \infty} \left[ M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right]^{p_k} = 0. \text{ Since } 0 < \inf p_k \leq p_k \leq 1$$

$$\lim_{k \rightarrow \infty} \left[ M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right] \leq \lim_{k \rightarrow \infty} \left[ M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right]^{p_k} = 0,$$

and hence  $x \in c_0(\Delta^n, M)$ .

**(ii)** Let  $p_k \geq 1$  for each  $k$  and  $\sup_k p_k < \infty$ . Let  $x \in c_0(\Delta^n, M)$ , then for each  $\varepsilon (0 < \varepsilon < 1)$  there exists a positive integer  $N$  such that

$$\left[ M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right] \leq \varepsilon, \quad \forall k \geq N.$$

Since  $1 \leq p_k \leq \sup p_k < \infty$ , we have

$$\lim_{k \rightarrow \infty} \left[ M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right]^{p_k} \leq \lim_{k \rightarrow \infty} \left[ M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right] \leq \varepsilon < 1.$$

Therefore  $x \in c_0(\Delta^n, M, p)$ . This completes the proof of the theorem.

**Theorem 2.5** Let  $M$  be Orlicz function that satisfy  $\Delta_2$ -condition. Then

$c_0(\Delta^n, M, p) \subset c(\Delta^n, M, p) \subset l_\infty(\Delta^n, M, p)$  and the inclusions are strict.

**Proof.** Let  $x \in c(\Delta^n, M, p)$ . We have

$$\begin{aligned} \left[ M \left( \frac{|\Delta^n x_k|}{\rho} \right) \right]^{p_k} &\leq C \left[ M \left( \frac{|\Delta^n x_k - l|}{\rho} \right) \right]^{p_k} + C \left[ M \left( \frac{|l|}{\rho} \right) \right]^{p_k} \\ &\leq C \left[ M \left( \frac{|\Delta^n x_k - l|}{\rho} \right) \right]^{p_k} + C \left[ K \delta^{-1} \left( \frac{|l|}{\rho} \right) M(2) \right]^H \end{aligned}$$

Thus we get  $x \in l_\infty(\Delta^n, M, p)$ . The inclusion  $c_0(\Delta^n, M, p) \subset c(\Delta^n, M, p)$  is obvious. To show that the inclusion is strict, consider the following example.

**Example 1 :** Let  $M(x) = x$  and  $p_k = 1$  for all  $k \in \mathbb{N}$ . Then the sequence  $x = ((-1)^k)$  belongs to  $l_\infty(\Delta^n, M, p)$ , but does not belong to  $c(\Delta^n, M, p)$ .

The proof of the following result is a routine work.

**Theorem 2.6** Let  $n \geq 1$ , then  $c(\Delta^{n-1}, M, p) \subset c_0(\Delta^n, M, p)$ .

**Theorem 2.7** The spaces  $c_0(M, p)$  and  $l_\infty(M, p)$  are solid and as such are monotone.

**Proof.** Let  $x \in c_0(M, p)$ . Then there exists  $\rho > 0$  such that  $\lim_{k \rightarrow \infty} \left[ M \left( \frac{|\Delta x_k|}{\rho} \right) \right] = 0$ .

Let  $(\alpha_k)$  be a sequence of scalars such that  $|\alpha_k| \leq 1$ . Then we have,

$$\left[ M \left( \frac{|\alpha_k x_k|}{\rho} \right) \right]^{p_k} \leq C \left[ M \left( \frac{|x_k|}{\rho} \right) \right]^{p_k}.$$

From the inequality it follows that  $c_0(M, p)$  is also solid. The monotonicity of the spaces  $c_0(M, p)$  and  $l_\infty(M, p)$  follows from Lemma 1.

The spaces  $c_0(\Delta^n, M, p)$ ,  $c(\Delta^n, M, p)$  and  $l_\infty(\Delta^n, M, p)$  are not solid in general. For showing that the above spaces are not solid, consider the following example.

**Example 2.** Let  $M(x) = x$  and  $p_k = 1$ , for all  $k \in \mathbb{N}$ . Consider the sequence

$x = (k^m)$ , for all  $k \in \mathbb{N}$ , then  $x \in l_\infty(\Delta^n, M, p)$ . Let  $(\alpha_k) = ((-1)^k)$ , then  $(\alpha_k x_k) \notin l_\infty(\Delta^n, M, p)$ .

**Corollary.**  $c_0(\Delta^n, M, p)$  and  $c(\Delta^n, M, p)$  are nowhere dense subsets of  $l_\infty(\Delta^n, M, p)$ .

**Theorem 2.8** The spaces  $c_0(\Delta^n, M, p)$ ,  $c(\Delta^n, M, p)$  and  $l_\infty(\Delta^n, M, p)$  are not sequence algebra, for  $n \geq 2$ .

**Proof.** Let  $M(x) = x$  and  $p_k = 1$ , for all  $k \in \mathbb{N}$ . Consider the sequences

$x = (k^m)$  and  $y = (k)$ , for all  $k \in \mathbb{N}$ , then  $x, y \in l_\infty(\Delta^n, M, p)$  but  $x \cdot y \notin l_\infty(\Delta^n, M, p)$ .

### 3. Particular Cases

If we consider the sequence spaces  $X(\Delta^n, M)$  instead of the paranormed sequence spaces  $X(\Delta^n, M, p)$ , then the spaces  $X(\Delta^n, M)$  will be a Banach spaces normed by

$$\|x\|_{\Delta^n} = \inf \left\{ \rho > 0 : \sup_{k \geq 0} M \left( \frac{|\Delta^n x_k|}{\rho} \right) \leq 1 \right\}.$$

The spaces  $X(M)$  will be a solid and monotone space. Further most of the results proved in the previous section will be true for this space too.

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