

## Half-lightlike submanifolds with planar normal sections in $R_2^4$

Feyza Esra ERDOĞAN<sup>1</sup>, Rıfat GÜNEŞ<sup>2</sup>, Bayram ŞAHİN<sup>2,\*</sup>

<sup>1</sup>Adıyaman University, Faculty of Education, Department of Mathematics, Adıyaman, Turkey

<sup>2</sup>İnönü University, Faculty of Science and Art, Department of Mathematics, Malatya, Turkey

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**Abstract:** We investigate half-lightlike submanifolds with planar normal sections of 4-dimensional pseudo-Euclidean space. We obtain necessary and sufficient conditions for a half-lightlike submanifold of  $R_2^4$  such that it has degenerate or nondegenerate planar normal sections.

**Key words:** Half-lightlike submanifold, planar normal sections

### 1. Introduction

Surfaces with planar normal sections in Euclidean spaces were first studied by Chen [1]. In [6], Y.H. Kim studied Surfaces with planar normal sections in semi-Riemann setting. As far as we know, however, this topic has not been studied in lightlike geometry. Therefore, as a first step, in this paper we study half-lightlike submanifolds with planar normal sections in  $R_2^4$ .

Let  $M$  be a hypersurface in  $R_2^4$ . For a point  $p$  in  $M$  and a lightlike vector  $\xi$  tangent to  $M$  at  $p$  that spans radical distribution, the vector  $\xi$  and transversal space  $tr(TM)$  to  $M$  at  $p$  determine a 2-dimensional subspace  $E(p, \xi)$  in  $R_2^4$  through  $p$ . The intersection of  $M$  and  $E(p, \xi)$  gives a lightlike curve  $\gamma$  in a neighborhood of  $p$ , which is called the normal section of  $M$  at the point  $p$  in the direction of  $\xi$ . Let  $v$  be a spacelike vector tangent to  $M$  at  $p$  ( $v \in S(TM)$ ). The vector  $v$  and transversal space  $tr(TM)$  to  $M$  at  $p$  then determine a 2-dimensional subspace  $E(p, v)$  in  $R_2^4$  through  $p$ . In this case, the intersection of  $M$  and  $E(p, v)$  gives a spacelike curve  $\gamma$  in a neighborhood of  $p$  which is called the normal section of  $M$  at  $p$  in the direction of  $v$ . According to both situations given above,  $M$  is said to have degenerate pointwise and nondegenerate pointwise planar normal sections, respectively, if each normal section  $\gamma$  at  $p$  satisfies  $\gamma' \wedge \gamma'' \wedge \gamma''' = 0$  [1,7,5,4].

### 2. Preliminaries

The codimension 2 lightlike submanifold  $(M, g)$  is called a half-lightlike submanifold if  $\text{rank}(radTM) = 1$ . In this case, there exist 2 complementary nondegenerate distributions  $S(TM)$  and  $S(TM^\perp)$  of  $RadTM$  in  $TM$  and  $TM^\perp$  respectively, called the screen and coscreen distribution on  $M$ . Then we have the following 2 orthogonal decompositions:

$$TM = RadTM \oplus_{orth} S(TM), TM^\perp = RadTM \oplus_{orth} S(TM^\perp),$$

where the symbol  $\oplus_{orth}$  denotes the orthogonal direct sum.

\*Correspondence: bayram.sahin@inonu.edu.tr

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We know from [2] that, for any smooth null section  $\xi$  of  $RadTM$  on a coordinate neighborhood  $U \subset M$ , there exists a uniquely defined null vector field  $N \in \Gamma(ltrTM)$  satisfying

$$\bar{g}(N, \xi) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \forall X \in \Gamma(S(TM)).$$

We call  $N$ ,  $ltr(TM)$ , and  $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$  the lightlike transversal vector field, lightlike transversal bundle, and transversal vector bundle of  $M$  with respect to the screen  $S(TM)$ , respectively. Since  $RadTM$  is a 1-dimensional vector subbundle of  $TM^\perp$  we may consider a supplementary distribution  $D$  to  $RadTM$  such that it is locally represented by  $u$ .

We call  $D$  a screen transversal bundle of  $M$ . Thus, we say that the vector bundle  $tr(TM)$  is defined over  $M$  by

$$tr(TM) = D \oplus_{orth} Itr(TM).$$

Therefore:

$$\begin{aligned} T\bar{M} &= S(TM) \perp (RadTM \oplus tr(TM)) \\ &= S(TM) \perp D \perp (RadTM \oplus Itr(TM)). \end{aligned} \tag{2.1}$$

Denote by  $P$  the projection of  $TM$  on  $S(TM)$  with respect to the decomposition (2.1). Then we write

$$X = PX + \eta(X)\xi, \quad \forall X \in \Gamma(TM),$$

where  $\eta$  is a local differential 1-form on  $M$  defined by  $\eta(X) = g(X, N)$ . Suppose  $\bar{\nabla}$  is the metric connection on  $\bar{M}$ . Since  $\{\xi, N\}$  is locally a pair of lightlike sections on  $U \subset M$ , we define symmetric  $F(M)$ -bilinear forms  $D_1$  and  $D_2$  and 1-forms  $\rho_1, \rho_2, \varepsilon_1$ , and  $\varepsilon_2$  on  $U$ . Using (2.1), we put

$$\bar{\nabla}_X Y = \nabla_X Y + D_1(X, Y)N + D_2(X, Y)u \tag{2.2}$$

$$\bar{\nabla}_X N = -A_N X + \rho_1(X)N + \rho_2(X)u \tag{2.3}$$

$$\bar{\nabla}_X u = -A_u X + \varepsilon_1(X)N + \varepsilon_2(X)u \tag{2.4}$$

for any  $X, Y \in \Gamma(TM)$ , where  $\nabla_X Y$ ,  $A_N X$ , and  $A_u X$  belong to  $\Gamma(TM)$ . We call  $D_1$  and  $D_2$  the lightlike second fundamental form and screen second fundamental form of  $M$  with respect to  $tr(TM)$ , respectively. Both  $A_N$  and  $A_u$  are linear operators on  $\Gamma(TM)$ . The first one is  $\Gamma(S(TM))$ -valued, called the shape operator of  $M$ . Since  $u$  is a unit vector field, (2.4) implies  $\varepsilon_2(X) = 0$ . In a similar way, since  $\xi$  and  $N$  are lightlike vector fields, from (2.2)–(2.4) we obtain

$$D_1(X, \xi) = 0, \quad \bar{g}(A_N X, N) = 0, \quad \bar{g}(A_u X, Y) = \epsilon D_2(X, Y) + \varepsilon_1(X)\eta(Y), \tag{2.5}$$

$$\varepsilon_1(X) = -\epsilon D_2(X, \xi), \quad \forall X \in \Gamma(TM). \tag{2.6}$$

Next, consider the decomposition (2.1), and then we have

$$\nabla_X PY = \nabla_X^* PY + E_1(X, PY)\xi, \tag{2.7}$$

$$\nabla_X \xi = -A_\xi^* X + u_1(X)\xi, \tag{2.8}$$

where  $\nabla_X^* PY$  and  $A_\xi^*$  belong to  $\Gamma(S(TM))$ .  $A_\xi^*$  is a linear operator on  $\Gamma(TM)$  and  $\nabla^*$  is a metric connection on  $S(TM)$ . We call  $E_1$  the local second fundamental form of  $S(TM)$  with respect to  $Rad(TM)$  and  $A_\xi^*$  the

shape operator of the screen distribution. The geometric object from Gauss and Weingarten equations (2.2)–(2.4) on one side and (2.7) and (2.8) on the other side are related by

$$\begin{aligned} E_1(X, PY) &= g(A_N X, PY), D_1(X, PY) = g(A_\xi^* X, PY), \\ u_1(X) &= -\rho_1(X), A_\xi^* \xi = 0, \end{aligned} \quad (2.9)$$

for any  $X, Y \in \Gamma(TM)$ . A half-lightlike submanifold  $(M, g)$  of a semi-Riemannian manifold  $(\bar{M}, \bar{g})$  is said to be totally umbilical in  $\bar{M}$  if there is a normal vector field  $\hat{Z} \in \Gamma(\text{tr}(TM))$  on  $M$ , called an affine normal curvature vector field of  $M$ , such that

$$h(X, Y) = D_1(X, Y)N + D_2(X, Y)u = \hat{Z}\bar{g}(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

In particular,  $(M, g)$  is said to be totally geodesic if its second fundamental form  $h(X, Y) = 0$  for any  $X, Y \in \Gamma(TM)$ . By direct calculation it is easy to see that  $M$  is totally geodesic if and only if both the lightlike and the screen second fundamental tensors  $D_1$  and  $D_2$  respectively vanish on  $M$ . Moreover, from (2.3), (2.5), (2.6), and (2.9) we obtain

$$A_\xi = A_u = \varepsilon_1 = \rho_2 = 0.$$

The notion of screen locally conformal half-lightlike submanifolds was introduced by Duggal and Sahin [3] as follows.

A half-lightlike submanifold  $M$ , of a semi-Riemannian manifold, is called screen locally conformal if on any coordinate neighborhood  $U$  there exists a nonzero smooth function  $\varphi$  such that for any null vector field  $\xi \in \Gamma(TM^\perp)$  the relation

$$A_N X = \varphi A_\xi^* X, \quad \forall X \in \Gamma(TM|_U) \quad (2.10)$$

holds between the shape operators  $A_N$  and  $A_\xi^*$  of  $M$  and  $S(TM)$ , respectively [3].

On the other hand, the notion of minimal half-lightlike submanifolds has been defined by Bejancu and Duggal as follows.

**Definition 2.1** *Let  $M$  be a half-lightlike submanifold of a semi-Riemannian manifold  $\bar{M}$ . We then say that  $M$  is a minimal half-lightlike submanifold if  $(\text{tr}|_{S(TM)} h = 0)$  and  $\varepsilon_1(X) = 0$  [3].*

**Definition 2.2** *A half-lightlike submanifold  $M$  is said to be irrotational if  $\bar{\nabla}_X \xi \in \Gamma(TM)$  for any  $X \in \Gamma(TM)$ , where  $\xi \in \Gamma(\text{Rad}TM)$  [3].*

For a half-lightlike  $M$ , since  $D_1(X, \xi) = 0$ , the above definition is equivalent to  $D_2(X, \xi) = 0 = \varepsilon_1(X)$ ,  $\forall X \in \Gamma(TM)$ .

### 3. Planar normal sections of half-lightlike hypersurfaces in $R_2^4$

In this section we consider half-lightlike submanifolds having planar normal section. First, we consider degenerate planar normal sections.

**3.1. Degenerate planar normal sections in half-lightlike submanifolds**

Let  $M$  be a half-lightlike submanifold in  $R_2^4$ . Now we investigate the conditions for a half-lightlike submanifold of  $R_2^4$  to have degenerate planar normal sections.

**Theorem 3.1** *Let  $M$  be a half-lightlike submanifold in  $R_2^4$ . Then  $M$  has degenerate planar normal sections if and only if*

$$D_2(\xi, \xi)u \wedge \bar{\nabla}_\xi D_2(\xi, \xi)u = 0, \tag{3.1.1}$$

where  $D_2$  is the screen second fundamental form of  $M$ .

**Proof** If  $\gamma$  is a null curve, for a point  $p$  in  $M$ , we have

$$\gamma'(s) = \xi, \tag{3.1.2}$$

$$\gamma''(s) = \nabla_\xi \xi + D_2(\xi, \xi)u, \tag{3.1.3}$$

$$\begin{aligned} \gamma'''(s) &= \nabla_\xi \nabla_\xi \xi + D_2(\nabla_\xi \xi, \xi)u \\ &\quad + \xi(D_2(\xi, \xi))u + D_2(\xi, \xi)(-A_u \xi + \varepsilon_1(\xi)N). \end{aligned} \tag{3.1.4}$$

From the definition of planar normal section and using  $Rad(TM) = Sp\{\xi\}$ , we get

$$\nabla_\xi \xi \wedge \xi = 0 \text{ and } \nabla_\xi \nabla_\xi \xi \wedge \xi = 0. \tag{3.1.5}$$

Assume that  $M$  has planar degenerate normal sections. Then

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0. \tag{3.1.6}$$

Thus, by using (3.1.2)–(3.1.5) in (3.1.6) one can see that  $D_2(\xi, \xi)u$  and  $D_2(\nabla_\xi \xi, \xi)u + \xi(D_2(\xi, \xi))u - D_2(\xi, \xi)A_u \xi + D_2(\xi, \xi)\varepsilon_1(\xi)N$  are linearly dependent. Taking the derivative of  $D_2(\xi, \xi)u$ , we obtain

$$\bar{\nabla}_\xi D_2(\xi, \xi)u = \xi(D_2(\xi, \xi))u - D_2(\xi, \xi)A_u \xi + D_2(\xi, \xi)\varepsilon_1(\xi)N,$$

where  $\gamma$  is assumed to be parameterized by a distinguished parameter. Hence, we get

$$D_2(\xi, \xi)u \wedge \bar{\nabla}_\xi D_2(\xi, \xi)u = 0.$$

Conversely, assume that  $D_2(\xi, \xi)u \wedge \bar{\nabla}_\xi D_2(\xi, \xi)u = 0$  for the degenerate tangent vector  $\xi$  of  $M$  at  $p$ . In this case, either  $D_2(\xi, \xi)u = 0$  or  $\bar{\nabla}_\xi D_2(\xi, \xi)u = 0$ . If  $D_2(\xi, \xi)u = 0$ , then  $M$  is totally geodesic in  $\bar{M}$  and  $M$  is totally umbilical. Thus, we obtain

$$\gamma'(s) = \xi, \tag{3.1.7}$$

$$\gamma''(s) = u_1(\xi)\xi, \tag{3.1.8}$$

$$\gamma'''(s) = \xi(u_1(\xi))\xi + u_1^2(\xi)\xi. \tag{3.1.9}$$

which give that  $M$  has degenerate planar normal sections. On the other hand, if  $\bar{\nabla}_\xi D_2(\xi, \xi)u = 0$ , then  $M$  is screen conformal. Hence, we have

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = \xi \wedge (\nabla_\xi \xi + D_2(\xi, \xi)u) \wedge (\nabla_\xi \nabla_\xi \xi + D_2(\xi, \nabla_\xi \xi)u + \bar{\nabla}_\xi D_2(\xi, \xi)u) = 0.$$

Hence, we complete the proof. □

Now we define a function

$$\begin{aligned} L_p : RadT_p M &\rightarrow R, \\ \xi &\rightarrow L_p(\xi) = D_2^2(\xi, \xi)\epsilon, \end{aligned}$$

where  $p \in M$  and  $\gamma(0) = p$ . If  $L_p(\xi) = D_2^2(\xi, \xi)\epsilon = 0$ , then we obtain  $D_2(\xi, \xi) = 0$  and  $\varepsilon_1(\xi) = 0$ . From (3.1.7), (3.1.8), and (3.1.9) we find  $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$ . Hence,  $M$  has degenerate planar normal sections.

We say that the curve  $\gamma$  has a vertex at the point  $p$  if the curvature  $\kappa$  of  $\gamma$  satisfies  $\frac{d\kappa^2(p)}{ds} = 0$  and  $\kappa^2 = \langle \gamma''(s), \gamma''(s) \rangle$ . Now let  $M$  have degenerate planar normal sections. Then  $L_p = 0$ , and so  $D_2(\xi, \xi) = 0$ . Hence, we get

$$h(\xi, \xi) = D_2(\xi, \xi)u = 0, (\bar{\nabla}_\xi h)(\xi, \xi) = 0,$$

which gives  $\bar{\nabla}h = 0$ . Moreover, we have

$$\epsilon\kappa^2(s) = \langle \gamma''(s), \gamma''(s) \rangle = 0$$

for any  $p \in M$ .

Consequently, we have the following result.

**Corollary 3.2** *Let  $M$  be a half-lightlike submanifold in  $R_2^4$  with degenerate planar normal sections such that*

$$\begin{aligned} L_p : RadT_p M &\rightarrow R, \\ \xi &\rightarrow L_p(\xi) = D_2^2(\xi, \xi)\epsilon, \end{aligned}$$

where  $p \in M$ . Then the following statements are equivalent:

1.  $D_2(\xi, \xi) = 0$ ,
2.  $(\bar{\nabla}_\xi h)(\xi, \xi) = 0$ ,
3.  $\bar{\nabla}h = 0$ ,
4. For any  $p \in M$ ,  $\kappa = 0$ .

Now, let us assume that a half-lightlike submanifold  $M$  of  $R_2^4$  has degenerate planar normal sections. Then for null vector  $\xi \in RadTM$ , we have

$$\nabla_\xi \xi \neq 0, \tag{3.1.10}$$

where  $\xi = \gamma'(s)$ , namely, the normal section  $\gamma$  is not a geodesic arc on a sufficiently small neighborhood of  $p$ . Then from (3.1.2)–(3.1.4) we write

$$\gamma'''(s) = a(s)\gamma''(s) + b(s)\gamma'(s),$$

where,  $a$  and  $b$  are differentiable functions for all  $p \in M$ . Hence, we get  $D_2(\xi, \xi) = \varepsilon_1(\xi) = 0$ .

Consequently, we have the following:

**Corollary 3.3** *Let a half-lightlike submanifold  $M$  in  $R_2^4$  have degenerate planar normal sections. If the normal section  $\gamma$  for any  $p$  is not a geodesic arc on a sufficiently small neighborhood of  $p$ , then  $D_2 = 0$  at  $RadTM$ .*

Next, assume that  $\gamma$  is parameterized by a distinguished parameter, namely,  $\gamma$  is a geodesic arc on a small neighborhood of  $p = \gamma(0)$ , i.e.  $\nabla_\xi \xi = 0$ . Since  $u_1(\xi) = \rho_1(\xi) = 0$ , we obtain

$$\begin{aligned} \gamma'(0) &= \xi, \\ \gamma''(0) &= D_2(\xi, \xi)u, \end{aligned} \tag{3.1.11}$$

$$\gamma'''(0) = \bar{\nabla}_\xi D_2(\xi, \xi)u = \xi(D_2(\xi, \xi))u - D_2(\xi, \xi)A_u\xi - \epsilon D_2^2(\xi, \xi)N. \tag{3.1.12}$$

Now, let us suppose that  $M$  has degenerate planar normal sections at  $\gamma(0) = p$ . Therefore, from  $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$ , we have  $\xi \wedge h(\xi, \xi) \wedge \bar{\nabla}_\xi h(\xi, \xi) = 0$ . From (3.1.11) and (3.1.12),  $\xi$ ,  $h(\xi, \xi)$ , and  $\bar{\nabla}_\xi h(\xi, \xi)$  are not linearly dependent. In this case, either  $h(\xi, \xi) = 0$  or  $\bar{\nabla}_\xi h(\xi, \xi) = 0$ . If  $\bar{\nabla}_\xi h(\xi, \xi) = 0$ , then we calculate

$$\begin{aligned} \langle h(\xi, \xi), h(\xi, w) \rangle &= -\langle \bar{\nabla}_\xi h(\xi, \xi), w \rangle \\ &= 0 \end{aligned} \tag{3.1.13}$$

and

$$\begin{aligned} \langle h(\xi, \xi), h(\xi, w) \rangle &= \langle h(\xi, \xi), \bar{\nabla}_w \xi \rangle - \langle h(\xi, \xi), \nabla_w \xi \rangle \\ &= \epsilon D_2(\xi, \xi) D_2(w, \xi). \end{aligned} \tag{3.1.14}$$

From the symmetry of bilinear forms  $D_1$  and  $D_2$  at  $\Gamma(TM)$ , hence from (3.1.13) and (3.1.14), we get  $D_2 = 0$  at  $\Gamma(TM)$ . Furthermore, from  $\bar{\nabla}_w \xi \in \Gamma(TM)$ , ( $\xi \in RadTM$ , and  $w \in \Gamma(TM)$ ), we see that  $M$  is irrotational. Then we have the following result.

**Corollary 3.4** *Let  $M$  be a half-lightlike submanifold of  $R_2^4$  with degenerate planar normal sections. If the normal section  $\gamma$  for any  $p$  is a geodesic arc on a sufficiently small neighborhood of  $p$ , then  $M$  is irrotational.*

Let  $M$  be a half-lightlike submanifold in  $R_2^4$  with degenerate planar normal sections. Since  $\gamma$  is a planar curve, we write

$$\gamma'''(s) = a(s)\gamma''(s) + b(s)\gamma'(s),$$

where  $a$  and  $b$  are differentiable functions for all  $p \in M$ . Then (3.1.8) gives

$$\begin{aligned} a(s) &= u_1(\xi) + \xi(\ln(D_2(\xi, \xi))), \\ b(s) &= \xi(u_1(\xi)) - D_2(\xi, \xi)\rho_2(\xi)\epsilon - u_1(\xi)\xi(\ln(D_2(\xi, \xi))). \end{aligned}$$

Moreover, we have  $\epsilon\kappa^2(s) = \langle \gamma''(s), \gamma''(s) \rangle = 0$  for any  $p \in M$ , which gives  $D_2(\xi, \xi) = \epsilon_1(\xi) = 0$ . Thus, we obtain

$$\begin{aligned} \gamma'''(s) &= u_1^2(\xi)\xi + u_1(\xi)D_2(\xi, \xi)u \\ &\quad + \xi(\ln(D_2(\xi, \xi)))D_2(\xi, \xi)u \\ &\quad + \xi(u_1(\xi))\xi - \epsilon D_2(\xi, \xi)\rho_2(\xi)\xi \end{aligned} \tag{3.1.15}$$

and

$$A_u \xi = \epsilon \rho_2(\xi) \xi. \tag{3.1.16}$$

Namely:

**Corollary 3.5** *Let  $M$  be a half-lightlike submanifold of  $R_2^4$  with degenerate planar normal sections, then  $A_u \xi$  is RadTM-valued.*

Now, from (3.1.15) and (3.1.16), we obtain

$$\begin{aligned} (\bar{\nabla}_\xi h)(\xi, \xi) &= \xi(\ln(D_2(\xi, \xi))) D_2(\xi, \xi) u \\ &\quad - \epsilon D_2(\xi, \xi) \rho_2(\xi) \xi - 2u_1(\xi) D_2(\xi, \xi) u. \end{aligned} \tag{3.1.17}$$

Let  $M$  be a half-lightlike submanifold of  $R_2^4$  with degenerate planar normal sections. If the normal section  $\gamma$  for any  $p$  is not a geodesic arc on a sufficiently small neighborhood of  $p$ , then we obtain

$$D_2(\xi, \xi) u \wedge (\bar{\nabla}_\xi h)(\xi, \xi) = 0. \tag{3.1.18}$$

Conversely, we assume that (3.1.18) is satisfied for any degenerate tangent vector  $\xi$  of  $M$ . Then either  $D_2(\xi, \xi) u = 0$  or  $(\bar{\nabla}_\xi h)(\xi, \xi) = 0$ . If  $D_2(\xi, \xi) u = 0$ , then from Theorem 3.1, we see that  $M$  has degenerate planar normal sections. On the other hand, if  $(\bar{\nabla}_\xi h)(\xi, \xi) = 0$ , then, by considering (3.1.5), we obtain

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = \xi \wedge D_2(\xi, \xi) u \wedge (\bar{\nabla}_\xi h)(\xi, \xi) = 0.$$

Consequently, we have the following:

**Corollary 3.6** *Let  $M$  be half-lightlike submanifold of  $R_2^4$  such that the normal section  $\gamma(s)$  for any  $p$  is not a geodesic arc on a sufficiently small neighborhood of  $p$ . Then half-lightlike submanifold  $M$  has planar normal sections if and only if (3.1.18) is satisfied.*

Now, let the normal section  $\gamma$  be a geodesic arc on a sufficiently small neighborhood of  $p$ , namely,  $\nabla_\xi \xi = 0 = u_1(\xi)$ . Since  $M$  has degenerate planar normal sections, we obtain

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = (\xi \wedge D_2(\xi, \xi) u \wedge D_2(\xi, \xi) A_u \xi) + (\xi \wedge D_2(\xi, \xi) u \wedge D_2(\xi, \xi) \varepsilon_1(\xi) N).$$

From Corollary 3.5, we have  $D_2(\xi, \xi) = 0$  and  $\varepsilon_1(\xi) = 0$ . Thus, we have the following result:

**Corollary 3.7** *Let  $M$  be a half-lightlike submanifold with degenerate planar normal section of  $R_2^4$ . The normal section  $\gamma$  for any  $p$  is a geodesic arc on a sufficiently small neighborhood of  $p$ . Then  $D_2(\xi, \xi) = 0$  or  $\varepsilon_1(\xi) = 0$ .*

Let  $M$  be a screen conformal half-lightlike submanifold of  $R_2^4(c)$  with degenerate planar normal sections. We denote the Riemann curvature tensors of  $\bar{M}$  and  $M$  by  $\bar{R}$  and  $R$ , and hence we have

$$\begin{aligned} \bar{g}(\bar{R}(X, Y) Z, PW) &= \varphi [D_1(X, Z) D_1(Y, PW) - D_1(Y, Z) D_1(X, PW)] \\ &\quad + \epsilon [D_2(X, Z) D_2(Y, PW) - D_2(Y, Z) D_2(X, PW)]. \end{aligned} \tag{3.1.19}$$

Let  $p \in M$  and  $\xi$  be a null vector of  $T_pM$ . A plane  $H$  of  $T_pM$  is called a null plane directed by  $\xi$  if it contains  $\xi$ ,  $\bar{g}(\xi, W) = 0$  for any  $W \in H$  and there exists  $W_0 \in H$  such that  $\bar{g}(W_0, W_0) \neq 0$ . Then the null sectional curvature of  $H$  with respect to  $\xi$  and  $\bar{\nabla}$  is defined by

$$K_\xi(H) = \frac{R_p(W, \xi, \xi, W)}{g_p(W, W)}. \tag{3.1.20}$$

Since  $v \in \Gamma(S(TM))$  and  $\xi \in \Gamma(RadTM)$ , we have

$$\begin{aligned} K_\xi(H) &= \varphi [D_1(v, \xi)D_1(\xi, v) - D_1(\xi, \xi)D_1(v, v)] \\ &\quad + \epsilon [D_2(v, \xi)D_2(\xi, v) - D_2(\xi, \xi)D_2(v, v)]. \end{aligned}$$

By using  $D_1(v, \xi) = 0$  in the last equation, we obtain

$$K_\xi(H) = \epsilon [D_2(v, \xi)D_2(\xi, v) - D_2(\xi, \xi)D_2(v, v)]. \tag{3.1.21}$$

Consequently, we have the following:

**Corollary 3.8** *Let  $M$  be a screen conformal half-lightlike submanifold of  $R_2^4(c)$  with degenerate planar normal sections. If  $M$  is minimal, then  $K_\xi(H) = 0$ .*

**Example 3.9** *Consider a surface  $M$  in  $R_2^4$  given by the equation*

$$x^3 = \frac{1}{\sqrt{2}}(x^1 + x^2); \quad x^4 = \frac{1}{2} \log \left( 1 + (x^1 - x^2)^2 \right).$$

*It is easy to see that  $M$  is a totally umbilical half-lightlike submanifold of  $R_2^4$ . Then by straightforward calculations we obtain*

$$D_2(\xi, \xi) = 0.$$

*Therefore, the intersection of  $M$  and  $E(p, \xi)$  gives a lightlike curve  $\gamma$  in a neighborhood of  $p$ , which is called the normal section of  $M$  at point  $p$  in the direction of  $\xi$ , namely*

$$\begin{aligned} \gamma'(s) &= \xi, \\ \gamma''(s) &= \bar{\nabla}_\xi \xi = 0. \end{aligned}$$

*Hence, we obtain*

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0.$$

### 3.2. Nondegenerate planar normal sections in half-lightlike submanifolds

In this subsection we investigate the conditions for a screen conformal half-lightlike submanifold  $M$  of  $R_2^4$  to have nondegenerate planar normal sections.

**Theorem 3.10** *Let  $M$  be a screen conformal half-lightlike submanifold in  $R_2^4$ .  $M$  has spacelike planar normal sections if and only if*

$$T(v, v) \wedge \bar{\nabla}_v T(v, v) = 0, \tag{3.2.1}$$

*where  $v \in \Gamma(S(TM))$  and  $T(v, v) = E_1(v, v)\xi + D_1(v, v)N + D_2(v, v)u$ .*

**Proof** Let  $M$  be a screen conformal half-lightlike submanifold and  $\gamma$  a spacelike curve on  $M$ . Then we have

$$\gamma'(s) = v, \tag{3.2.2}$$

$$\gamma''(s) = \bar{\nabla}_v v = \nabla_v^* v + E_1(v, v)\xi + D_1(v, v)N + D_2(v, v)u, \tag{3.2.3}$$

$$\begin{aligned} \gamma'''(s) = & \nabla_v^* \nabla_v^* v + E_1(v, \nabla_v^* v)\xi + D_1(v, \nabla_v^* v)N + D_2(v, \nabla_v^* v)u \\ & + v(E_1(v, v))\xi + v(D_1(v, v))N + v(D_2(v, v))u \\ & - E_1(v, v)A_\xi^* v + E_1(v, v)u_1(v)\xi + E_1(v, v)D_2(v, \xi)u \\ & - D_1(v, v)A_N v + D_1(v, v)\rho_1(v)N + D_1(v, v)\rho_2(v)u \\ & - D_2(v, v)A_u v + D_2(v, v)\varepsilon_1(v)N, \end{aligned} \tag{3.2.4}$$

where  $\nabla^*$  is the induced connection of  $M'$  and  $\gamma'(s) = v$ ,  $\gamma'(0) = v$ . From the definition of a planar normal section and  $S(TM) = Sp\{v\}$  we have

$$v \wedge \nabla_v^* v = 0 \text{ and } v \wedge \nabla_v^* \nabla_v^* v = 0. \tag{3.2.5}$$

Assume that  $M$  has planar nondegenerate normal sections. Then we have

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0.$$

Thus, from (3.2.5),

$$T(v, v) = E_1(v, v)\xi + D_1(v, v)N + D_2(v, v)u$$

and

$$\begin{aligned} \bar{\nabla}_v T(v, v) = & E_1(v, \nabla_v^* v)\xi + D_1(v, \nabla_v^* v)N + D_2(v, \nabla_v^* v)u \\ & + v(E_1(v, v))\xi + v(D_1(v, v))N + v(D_2(v, v))u \\ & - E_1(v, v)A_\xi^* v + E_1(v, v)u_1(v)\xi + E_1(v, v)D_2(v, \xi)u \\ & - D_1(v, v)A_N v + D_1(v, v)\rho_1(v)N + D_1(v, v)\rho_2(v)u \\ & - D_2(v, v)A_u v + D_2(v, v)\varepsilon_1(v)N \end{aligned}$$

are linearly dependent, where  $\gamma$  is assumed to be parameterized by arc length. Thus, we obtain

$$T(v, v) \wedge \bar{\nabla}_v T(v, v) = 0.$$

Conversely, we assume that  $T(v, v) \wedge \bar{\nabla}_v T(v, v) = 0$  for a spacelike tangent vector  $v$  of  $M$  at  $p$ . Then either  $T(v, v) = 0$  or  $\bar{\nabla}_v T(v, v) = 0$ . If  $T(v, v) = 0$ , then from (3.2.2), (3.2.3), (3.2.4), and (3.2.5),  $M$  has degenerate planar normal sections. If  $\bar{\nabla}_v T(v, v) = 0$ , from (3.2.5), we obtain

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = v \wedge T(v, v) \wedge \bar{\nabla}_v T(v, v) = 0.$$

□

**Example 3.11** Let  $M$  be a half-lightlike submanifold of the 4-dimensional semi-Riemann space  $(R_2^4, \bar{g})$  of index 2, as given in Example 3.9. Now, for a point  $p$  in  $M$  and a spacelike vector  $U_2$  tangent to  $M$  at  $p$  ( $U_2 \in S(TM)$ ), the vector  $U_2$  and transversal space  $tr(TM)$  to  $M$  at  $p$  determine a 2-dimensional subspace  $E(p, U_2)$  in  $R_2^4$  through  $p$ . The intersection of  $M$  and  $E(p, U_2)$  gives a spacelike curve  $\gamma$  in a neighborhood of  $p$ . Now we research half-lightlike submanifolds of the  $R_2^4$  semi-Riemannian manifold to have the condition of nondegenerate planar normal sections. Hence, we obtain

$$\begin{aligned} U_1 &= \sqrt{2} \left(1 + (x^1 - x^2)^2\right) \partial_1 + \left(1 + (x^1 - x^2)^2\right) \partial_3 + \sqrt{2} (x^1 - x^2) \partial_4, \\ U_2 &= \sqrt{2} \left(1 + (x^1 - x^2)^2\right) \partial_1 + \left(1 + (x^1 - x^2)^2\right) \partial_3 - \sqrt{2} (x^1 - x^2) \partial_4, \\ \xi &= \partial_1 + \partial_2 + \sqrt{2} \partial_3, \\ u &= 2(x^2 - x^1) \partial_2 + \sqrt{2} (x^2 - x^1) \partial_3 + (1 + (x^1 - x^2)) \partial_4. \end{aligned}$$

$$N = -\frac{1}{2} \partial_1 + \frac{1}{2} \partial_2 + \frac{1}{\sqrt{2}} \partial_3$$

and

$$\begin{aligned} \gamma' &= U_2 = \sqrt{2} \left(1 + (x^1 - x^2)^2\right) \partial_1 + \left(1 + (x^1 - x^2)^2\right) \partial_3 - \sqrt{2} (x^1 - x^2) \partial_4, \\ \gamma'' &= 2 \left(1 + (x^1 - x^2)^2\right) \cdot \left\{ 2(x^2 - x^1) \partial_2 + \sqrt{2} (x^2 - x^1) \partial_3 + \partial_4 \right\}, \\ \gamma''' &= \sqrt{2} \left(1 + (x^1 - x^2)^2\right) \left[ \begin{aligned} &4 \left(1 + 3 \left((x^1 - x^2)^2\right)\right) \partial_2 \\ &+ 2\sqrt{2} \left(1 + 3 \left((x^1 - x^2)^2\right)\right) \partial_3 - 4(x^1 - x^2) \partial_4 \end{aligned} \right] \\ &+ \frac{4\sqrt{2} (x^2 - x^1)^3}{\left(1 + (x^1 - x^2)^4\right)} \left(1 + (x^1 - x^2)^2\right) \left[ \begin{aligned} &2(x^2 - x^1) \partial_2 \\ &+ \sqrt{2} (x^2 - x^1) \partial_3 + (1 + x^1 - x^2) \partial_4 \end{aligned} \right]. \end{aligned}$$

Then, by direct calculations we find

$$E_1(U_2, U_2) = 0, \tag{3.2.6}$$

$$E_1(U_2, \nabla_{U_2}^* U_2) = 0. \tag{3.2.7}$$

Thus, from (3.2.6) and (3.2.7),  $T(U_2, U_2)$  and  $\bar{\nabla}_{U_2} T(U_2, U_2)$  are linearly dependent. Hence we have  $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$ .

**Proposition 3.12** Let  $M$  be a half-lightlike submanifold in  $R_2^4$ . If  $M$  has planar normal sections, then

$$\nabla_v^* v = 0, \tag{3.2.8}$$

where  $\gamma$  is a normal section in the direction  $v = \gamma'(s)$  for  $v \in \Gamma(S(TM))$ .

**Proof** From  $v \in S(TM)$  we have

$$\langle v, v \rangle = 1 \Rightarrow \langle v, \nabla_v^* v \rangle = 0. \tag{3.2.9}$$

Using the definition of a normal section and (3.2.9), we complete the proof. □

Now we define a function  $L$  by

$$L(p, v) = L_p(v) = \langle T(v, v), T(v, v) \rangle$$

on  $\bigcup_p M$ , where  $\bigcup_p M = \left\{ v \in \Gamma(TM) \mid \langle v, v \rangle^{\frac{1}{2}} = 1 \right\}$ . If  $L \neq 0$ , then  $M$  has nondegenerate pointwise normal sections. By a vertex of curve  $\gamma$  we mean a point  $p$  on  $\gamma$  such that its curvature  $\kappa$  satisfies  $\frac{d\kappa^2(0)}{ds} = 0$ . Let  $M$  have planar normal sections. From Proposition 3.12 we obtain

$$\begin{aligned} \epsilon \kappa^2(s) &= 2E_1(v, v) D_1(v, v) + D_2^2(v, v) \epsilon, \\ \frac{1}{2} \frac{d\kappa^2(0)}{ds} &= v(E_1(v, v) D_1(v, v)) + v(D_2(v, v)) D_2(v, v) \epsilon. \end{aligned}$$

If  $M$  is totally geodesic, then  $D_1 = D_2 = 0$ . Thus  $\gamma$  has a vertex. Consequently, we have the following:

**Corollary 3.13** *Let  $M$  be a half-lightlike submanifold of  $R_2^4$ . If  $M$  has nondegenerate planar normal sections the submanifold is totally geodesic screen conformal at  $p \in M$ , if and only if normal section curve  $\gamma$  has a vertex at  $p \in M$ .*

**Corollary 3.14** *Let  $M$  be a half-lightlike submanifold of  $R_2^4$  with planar normal sections. Then normal section curve  $\gamma$  has a vertex and the submanifold is totally geodesic if and only if  $M$  is minimal.*

**Proof** If  $M$  is totally geodesic, then from  $(tr|_{S(TM)} h = 0)$  and  $\epsilon_1(\xi) = 0$ , we conclude.

From Corollary 8 and Corollary 9, we give: □

**Corollary 3.15** *Let  $M$  be a half-lightlike submanifold in  $R_2^4(c)$  with planar normal sections. Then  $K_\xi(H) = 0$  if and only if normal section curve  $\gamma$  has a vertex at  $p \in M$  where  $\xi \in \Gamma(RadTM)$ .*

**Corollary 3.16** *Let  $M$  be a half-lightlike submanifold of  $R_2^4$  and the normal section  $\gamma$  at for any  $p$  be a geodesic arc on a sufficiently small neighborhood of  $p$ . Then  $M$  has nondegenerate planar normal sections if and only if*

$$h(v, v) \wedge (\bar{\nabla}_v h)(v, v) = 0,$$

where is  $h(v, v) = D_1(v, v)N + D_2(v, v)u$ .

**Proof** If normal section  $\gamma$  at for any  $p$  is a geodesic arc on a sufficiently small neighborhood of  $p$ , we have

$$\begin{aligned} \gamma'(s) &= v \\ \gamma''(s) &= D_1(v, v)N + D_2(v, v)u \\ \gamma'''(s) &= v(D_1(v, v))N + v(D_2(v, v))u \\ &\quad - D_1(v, v)A_N v + D_1(v, v)\rho_1(v)N \\ &\quad + D_1(v, v)\rho_2(v)u - D_2(v, v)A_u v \\ &\quad + D_2(v, v)\epsilon_1(v)N. \end{aligned}$$

Therefore, by taking the covariant derivative of

$$h(v, v) = D_1(v, v)N + D_2(v, v)u,$$

we obtain

$$(\bar{\nabla}_v h)(v, v) = \bar{\nabla}_v h(v, v) = \gamma'''(s),$$

which gives

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = v \wedge h(v, v) \wedge (\bar{\nabla}_v h)(v, v) = 0.$$

From the last equation above, we have

$$h(v, v) \wedge (\bar{\nabla}_v h)(v, v) = 0.$$

Conversely, we assume that  $h(v, v) \wedge (\bar{\nabla}_v h)(v, v) = 0$ . In this case, we have either  $h(v, v) = 0$  or  $(\bar{\nabla}_v h)(v, v) = 0$ . If  $h(v, v) = 0$ , we have  $D_1(v, v) = 0$  and  $D_2(v, v) = 0$ . In this way, we get

$$\varepsilon_1(\xi) = 0,$$

which shows that  $M$  is minimal and has planar normal sections. On the other hand, if  $(\bar{\nabla}_v h)(v, v) = 0$ , from  $(\bar{\nabla}_v h)(v, v) = \bar{\nabla}_v h(v, v) = \gamma'''(s) = 0$ , we obtain

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0;$$

that is,  $M$  has nondegenerate planar normal sections. □

We also have the following result:

**Corollary 3.17** *Let  $M$  be a half-lightlike submanifold in  $R_2^4$  and the normal section  $\gamma$  for any  $p$  be a geodesic arc on a sufficiently small neighborhood of  $p$ . Then the following statements are equivalent:*

1.  $(\bar{\nabla}_v h)(v, v) = 0$ ,
2.  $\bar{\nabla}h = 0$ ,
3.  $M$  has nondegenerate planar normal sections of  $p \in M$  and  $\gamma$  has a vertex point at  $p \in M$ ,
4.  $D_2 = 0$  in  $S(TM)$ .

**Proof** For curvature  $\kappa$  at  $p$  point of  $\gamma$ , we have

$$\begin{aligned} \epsilon\kappa^2(s) &= D_2^2(v, v)\epsilon, \\ \frac{1}{2}\epsilon\frac{d\kappa^2(s)}{ds} &= v(D_2(v, v))D_2(v, v)\epsilon, \end{aligned} \tag{3.2.10}$$

and from  $\epsilon\kappa^2(s) = \langle \gamma''(s), \gamma''(s) \rangle$ ,

$$\begin{aligned} \frac{1}{2}\epsilon\frac{d\kappa^2(s)}{ds} &= \langle (\bar{\nabla}_v h)(v, v), h(v, v) \rangle \\ &= 0. \end{aligned} \tag{3.2.11}$$

Hence, from (3.2.10) and (3.2.11), we obtain  $D_2(v, v) = 0$ . From here, we complete the proof. □

**Example 3.18** Consider a surface  $M$  in  $R_1^4$  given by the equation

$$x_1 = x_3, x_2 = (1 - x_4)^{\frac{1}{2}}.$$

Then we obtain:

$$\begin{aligned} TM &= Sp\{\xi = \partial x_1 + \partial x_3, v = -x_4\partial x_2 + x_2\partial x_4\}, \\ TM^\perp &= Sp\{\xi = \partial x_1 + \partial x_3, u = x_2\partial x_2 + x_4\partial x_2\}. \end{aligned}$$

Therefore, we have  $RadTM = Sp\{\xi\}$ ,  $S(TM) = Sp\{v\}$ ,  $S(TM^\perp) = Sp\{u\}$ , and  $ltr(TM) = Sp\{N = \frac{1}{2}(\partial x_1 + \partial x_3)\}$ , which show that  $M$  is a half-lightlike submanifold of  $R_1^4$ . Then using the Gauss and Weingarten formulas and on account of  $\nabla_v\xi = 0$ , we have

$$\bar{g}(A_\xi^*v, v) = 0 \Rightarrow A_\xi^*v = 0.$$

Moreover, from (2.3) and by straightforward calculations we obtain

$$A_Nv = 0$$

or  $A_Nv \in RadTM$ . Using (2.2), we have

$$D_1(v, v) = \bar{g}(A_\xi^*v, v) = 0,$$

and since  $D_1(v, \xi) = 0$ , we have  $D_1 = 0$ . Using (2.2), we obtain

$$D_2(v, v)\epsilon = \bar{g}(v, A_uv).$$

Since

$$\bar{g}(\bar{\nabla}_v\bar{\nabla}_vv, N) = \bar{g}(A_uv, N) = 0,$$

$A_uv \in S(TM)$  and by straightforward calculations we obtain

$$\begin{aligned} \bar{g}(\bar{\nabla}_v\bar{\nabla}_vv, u) &= 2x_2x_4 \\ \bar{g}(\bar{\nabla}_v\bar{\nabla}_vv, \xi) &= -\epsilon_1(u) = 0 \\ \rho_1(v) &= -\bar{g}(A_Nv, \xi) = 0, \rho_2(v) = -\epsilon\bar{g}(A_Nv, u) = 0 \\ \epsilon_1(u) &= 0 \Rightarrow D_2(v, \xi) = 0, \epsilon D_2(v, v) = -1. \end{aligned}$$

Thus,  $M$  is a screen conformal totally umbilical half-lightlike submanifold. Let  $v \in S(TM)$  and  $p \in M$ . We denote subspace

$$E(p, v) = \{v\} \cup tr(TM)$$

and we have

$$E(p, v) \cap M = \gamma,$$

where  $\gamma$  is the normal section of  $M$  at  $p$  in the direction of  $v$ . Then we have

$$\begin{aligned} \gamma'(s) &= v = -x_4\partial x_2 + x_2\partial x_4 \\ \gamma''(s) &= \bar{\nabla}_vv = -2x_2\partial x_2 - 2x_4\partial x_4 \\ \gamma'''(s) &= \bar{\nabla}_v\bar{\nabla}_vv = 2x_4\partial x_2 - 2x_2\partial x_4. \end{aligned}$$

Hence,

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0;$$

that is,  $M$  has nondegenerate planar normal sections.

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